Tropical Ideals with Hilbert Function Two

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

We study tropical ideals with Hilbert function two, i.e., homogeneous ideals in the tropical polynomial semiring such that each degree has the underlying structure of a valuated matroid of rank two. We classify all such ideals when the matroid is over the field $\mathbb{B}$. We discuss the realizability of such ideals when they are in two and three variables. We also give necessary conditions for a sequence of valuated matroids over $\mathbb{R}$ to correspond to a tropical ideal in two variables.

Chapter 1

Introduction

Tropical algebraic geometry is an area of algebraic geometry that replaces classical varieties with their piecewise linear shadows. Namely, the tropicalization of a classical variety is a polyhedral complex which retains many of the invariants of the original variety such as, for example, the dimension. Due to its polyhedral structure, tropical varieties are usually easier to work with than their classical counterparts.

Tropical Mathematics emerged as a field in the second half of the 20th century when a computer scientist Imre Simon started using min-plus algebra. However, it attracted more attention from the mathematical community only in 2003 when Mikhalkin published his breakthrough work on a new way of computing Gromov-Witten invariants which was based on tropical geometry [9].

Since then, tropical geometry was successfully used to simplify problems in algebraic geometry, combinatorics, computational algebra and also beyond mathematics in, for example, economics.

Until recently, most of the work was focused on studying tropical varieties and cycles. However, in 2013 Giansiracusa and Giansiracusa introduced tropical schemes which aim to generalize constructions in scheme theory to semi-rings [4]. Their work was followed by Maclagan and Rincón who connected the previous work with valuated matroids [5] and introduced the notion of a tropical ideal [6], and by Macpherson who recovered the work done in [4] using non-archimedean geometry [8]. Having a tropical scheme theory would allow us to develop new tools to tropically solve problems of the classical algebraic geometry.

In the current work we take the definition of a tropical ideal as suggested by Maclagan and Rincón in [6]. Namely, we consider a subclass of homogeneous ideals in the tropical polynomial semiring which has the property that each homogeneous degree part has the underlying structure of a valuated matroid (Definition 2.4.1).

The set of these ideals contains tropicalization of all classical ideals, but is also strictly bigger than that. We have

$$\begin{aligned}\{\text{tropicalization of classical ideals}\} &\subsetneq \{\text{tropical ideals}\}\\ &\subsetneq \{\text{ideals in the tropical polynomial semiring}\}.\end{aligned}$$

However, whilst ideals in the tropical polynomial semiring can give varieties which are not polyhedral complexes, the above definition gives us ideals whose varieties are finite polyhedral complexes. Unlike classical ideals, tropical ideals are not finitely generated. Moreover, due to their matroidal nature, it is often easier to describe them saying which polynomials they do not contain, rather than the opposite.

In this work we start by introducing the necessary language and the notation needed later. In Chapter 2 we give some background necessary to study tropical ideals. We also give examples of tropical ideals with suggestions how to think about them. In Chapter 3 we study tropical ideals with Hilbert function one. In particular, Theorem 3.0.2 states that all of them are tropicalizations of classical ideals over a point. This can also be found in [6].

Chapter 4 contains the main theorem, Theorem 4.2.4, which describes all tropical ideals without coefficients with Hilbert function two.

Theorem. *Let L be the lattice consisting of all integer points in $\mathbb{Z}^n$ in the hyperplane defined by $x_1 + x_2 + \cdots + x_n = 0$. There is a one to one correspondence between the set of all homogeneous tropical ideals with Hilbert function two, saturated with respect to $x_1x_2\cdots x_n$ and proper sublattices of L.*

In Chapter 5 we discuss realizability of tropical ideals with Hilbert function two and without coefficients. That is, for a tropical ideal J we study when there exists a classical ideal I such that $\text{trop}(I) = J$. In particular, we prove the following theorem (Theorem 5.1.5).

Theorem. *All tropical ideals without coefficients in two variables with Hilbert function two are realizable.*

After this, we will focus on a three variables case. It turns out that some tropical ideals in three variables are realizable only over specific fields. We will give examples of such ideals.

In Chapter 6 we focus on tropical ideals in two variables with Hilbert function two with coefficients. In particular, Proposition 6.1.10 together with Lemma 6.1.11 and Conjecture 6.1.12, give necessary conditions for a sequence of matroids to correspond to a tropical ideal with Hilbert function two. We finish Chapter 6 by stating a conjecture about the sufficiency of these conditions.

Chapter 2

Background

In this chapter we will introduce the background on Hilbert functions and matroid theory needed later. We will also define tropical ideals, the main objects in this work. At the end of this chapter we explain the conventions we will be using.

2.1 Hilbert function

The goal of this section is to describe the ideals in the polynomial ring $S = K[x_1, \dots, x_n]$ which have constant Hilbert function one or two. In this work we will assume familiarity with Gröbner basis theory at the same level as in [1]. We refer the reader to this book for the proofs of all the statements in this section.

The definition of Hilbert function we will be using is the following.

Definition 2.1.1. *Let K be a field and I a homogeneous ideal in the polynomial ring $S = K[x_1, \dots, x_n]$. The* Hilbert function *of I is the function on the non-negative integers i defined by $H_{S/I}(i) := \dim_K(S_i/I_i)$, where S_i, I_i are the homogeneous degree i parts of S, I, respectively.*

When we say that an ideal I has constant Hilbert function c we mean that $H_{S/I}(i) = c$ for all $i \geq 1$.

Definition 2.1.2. *Let $[a_1 : \dots : a_n]$ be a point in $\mathbb{P}^{n-1}_K$. An ideal $I_p \subset K[x_1, \dots, x_n]$ consisting of all polynomial functions vanishing at this point is called a* point ideal. *It is given by $I_p = \langle a_i x_j - a_j x_i \mid 1 \leq i, j \leq n \rangle$.*

We are ready to describe all ideals with Hilbert function one.

Theorem 2.1.3. *Let $p = [a_1 : \dots : a_n] \in \mathbb{P}^{n-1}$ and $S = K[x_1, \dots, x_n]$. Then $H_{S/I_p}(d) = 1$ for all $d \geq 0$ and any ideal with Hilbert function one in S is of the form I_p for some $p \in \mathbb{P}^{n-1}$.*

We will see in Chapter 3 that all tropical ideals with Hilbert function one are also determined by their degree one part.

Let us shift our attention to ideals with Hilbert function two. It turns out that all such ideals are determined by their degree one and two homogeneous parts. As a warm-up before we look at the tropical ideals, let us have a look at the following example.

Example 2.1.4. Let $I \subset K[x,y]$ be a (homogeneous) monomial ideal. If its degree one part has codimension two it means it is equal to $\{0\}$. The homogeneous degree two part must contain only one monomial from the set $\{x^2, xy, y^2\}$. If it contains the monomial x^2, then the degree d of I contains all monomials but $\{xy^{d-1}, y^d\}$. If it contains the monomial xy, then the degree d of I contains all monomials but $\{x^d, y^d\}$. Finally, if it contains the monomial y^2, then the degree d of I contains all monomials but $\{x^d, x^{d-1}y\}$. So the ideal I has Hilbert function two.

It is a general rule that every ideal in $K[x_1, \dots, x_n]$ with constant Hilbert function two is generated by $n-2$ linear polynomials and one polynomial of degree two.

Proposition 2.1.5. *If I is a homogeneous ideal in $K[x_1, \dots, x_n]$ with constant Hilbert function two, then its degree one and two parts describe it completely.*

We will see that this is not the case with the tropical ideals. Tropical ideals with Hilbert function two are not necessarily determined by their degree one and two parts.

One last fact from [1] that we need to introduce in this section, and which we will use later in Chapter 5, is the following.

Proposition 2.1.6. *Let I be an ideal in $K[x_1, \dots, x_n]$ and let G be a Gröbner basis of I under the lexicographic order with $x_n > \cdots > x_1$. Then, for every $1 \leq l \leq n$, the set*

$$G_l = G \cap K[x_1, \dots, x_l]$$

is a Gröbner basis for $I \cap K[x_1, \dots, x_l]$.

2.2 Matroids

Matroids are combinatorial objects which generalize the notion of linear independency. They can be defined in many cryptomorphically equivalent ways, for example, in term of bases, independent sets or circuits. In [2] and [3], Dress and

Wenzel extended the notion of a matroid to a valuated matroid. We will need valuated matroids to define a tropical ideal.

We will present below three different but equivalent definition of a matroid. The one we will give first is in terms of the independent sets.

Definition 2.2.1. *Let V be a finite set. A* matroid *M is a pair $(V, \mathcal{I})$ where $\mathcal{I}$ is a family of subsets of V, called the independent sets, with the following properties:*

(MI1) *the empty set is independent, i.e., $\emptyset \in \mathcal{I}$,*

(MI2) *every subset of an independent set is independent, i.e., for each $I' \subset I \subset V$, if $I \in \mathcal{I}$ then $I' \in \mathcal{I}$,*

(MIE) *if I_1 and I_2 are two independent sets of $\mathcal{I}$ and I_1 has more elements than I_2, then there exists an element in I_1 that when added to I_2 gives a larger independent set than I_2.*

A subset of V which is not independent is called dependent. An example of a matroid is a matroid M on the set $V = \{1, 2, 3, 4, 5\}$ where

$$\mathcal{I} = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}\,.$$

It is easy to check that all the axioms for independent sets are satisfied.

The following defines a matroid in terms of circuits.

Definition 2.2.2. *Let V be a finite set. A* matroid *is a pair $M = (V, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of V, called the circuits family, with the following properties:*

(MC1) *the empty set is not a circuit, i.e., $\emptyset \notin \mathcal{C}$,*

(MC2) *if C_1 and C_2 are two different circuits in $\mathcal{C}$ then $C_1 \nsubseteq C_2$,*

(MCE) *for $C_1, C_2 \in \mathcal{C}$, $u \in C_1 \cap C_2$ and $v \in C_1 - C_2$, there exists $C_3 \in \mathcal{C}$ such that $v \in C_3 \subseteq (C_1 \cup C_2) - u$,*

where $C_1 - C_2$ denotes the difference set $\{v \mid v \in C_1, v \notin C_2\}$ and $(C_1 \cup C_2) - u$ is a shorthand notation for $(C_1 \cup C_2) - \{u\}$.

Circuits and independent sets are related in the following way: a circuit in a matroid M is a minimal dependent subset of V, where a minimal dependent subset means that it is a dependent set whose proper subsets are all independent. In the example of a matroid above the corresponding family of circuits is

$$\mathcal{C} = \{\{5\}, \{1,2\}, \{1,3\}, \{2,3\}\}\,.$$

The last definition of a matroid we will give here is the following one.

Definition 2.2.3. *Let V be a finite set. A* matroid *is a pair $M = (V, \mathcal{B})$, where $\mathcal{B}$ is a family of subsets of V, called the base family, with the following properties:*

(BM1) *$\mathcal{B} \neq \emptyset$,*

(BMX) *for $B_1, B_2 \in \mathcal{B}$ and $u \in B_1 - B_2$, there exists $v \in B_2 - B_1$ such that $B_1 - u + v \in \mathcal{B}$ and $B_2 - v + u \in \mathcal{B}$,*

where for $B \in \mathcal{B}$ and $u, v \in V$, $B - u + v \in \mathcal{B}$ is a shorthand notation for $B \setminus \{u\} \cup \{v\}$.

The relation between independent sets and bases is that a basis is a maximal independent set, i.e., an independent set which becomes dependent on adding any element of V. Circuits and bases are related in the following way: a circuit is a minimal subset of V not belonging to any basis of M. For details on this and the proof that all the above definitions of a matroid are equivalent see, for example, [11].

In our running example the base family is

$$\mathcal{B} = \{\{1,4\}, \{2,4\}, \{3,4\}\}\,.$$

Axiom (MIE) implies that the maximal independent subsets of V are all equal in size. To see this, take two maximal subsets of V and apply (MIE) if they do not have the same size. It follows that every base of a matroid has the same cardinality. So the following notion is well defined.

Definition 2.2.4. *The* rank *of a matroid M is the cardinality of a base of M.*

It follows that in our example the rank of the matroid is two.

There is one class of matroids which we will see often in what follows. The *uniform matroid* $\mathcal{U}_n^r$ is a matroid on a set V of size n such that a subset of V is independent if and only if it contains at most r elements. It follows that all subsets of V of size r are bases and all circuits have size $r + 1$. The rank of the uniform matroid $\mathcal{U}_n^r$ is r.

One of the generalizations of the notion of a matroid are valuated matroids. Similarly to matroids, valuated matroids can also be defined in a few equivalent ways. Below we will give two descriptions: in terms of valuated circuits and vectors. In the next definition, if X is an element of $(\mathbb{R} \cup \{\infty\})^V$, where V is a finite set, X_i denotes the i-th element of the $|V|$-tuple and $\underline{X}$ denotes the support of X, that is, the set of entries in the $|V|$-tuple which are not ∞.

Definition 2.2.5. *Let V be a finite set. A* valuated matroid *on V is a family $\mathcal{X} \subseteq (\mathbb{R} \cup \{\infty\})^V$ such that*

(VC1) $(\infty, \dots, \infty) \notin \mathcal{X}$,

(VC2) *if $X, Y \in \mathcal{X}$ with $\underline{X} \neq \underline{Y}$ then $\underline{X} \not\subseteq \underline{Y}$,*

(VC3) *for $X \in \mathcal{X}$ and $\alpha \in \mathbb{R}$, $X + \alpha \mathbf{1} \in \mathcal{X}$ holds,*

(VCE) *for $X, Y \in \mathcal{X}$ and $u, v \in V$ with $X_u = Y_u \neq \infty$ and $X_v < Y_v$, there exists $Z \in \mathcal{X}$ such that $Z_u = \infty$, $Z_v = X_v$ and $Z \geq \min(X, Y)$.*

We call a member of $\mathcal{V}$ a valuated circuit of a valuated matroid and the conditions (VC1), (VC2), (VC3), (VCE) the valuated circuit axioms of a valuated matroid.

Before giving an example, we will introduce the definition of a valuated matroid in terms of valuated vectors.

Definition 2.2.6. *Let V be a finite set. A* valuated matroid *on V is a family of $|V|$-tuples $\mathcal{V} \subseteq (\mathbb{R} \cup \{\infty\})^V$ such that*

(VV1) $(\infty, \dots, \infty) \in \mathcal{V}$,

(VV2) *if $X, Y \in \mathcal{V}$, then $\min(X, Y) \in \mathcal{V}$, where the minimum is taken coordinate-wise,*

(VV3) *for $X \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, we have $X + \alpha \mathbf{1} \in \mathcal{V}$, where $\mathbf{1} = (1, \dots, 1)$,*

(VVE) *for $X, Y \in \mathcal{V}$ and $u \in V$ with $X_u = Y_u \neq \infty$, there is $Z \in \mathcal{V}$ such that $Z_u = \infty$, $Z \geq \min(X, Y)$ and $Z_i = \min(X_i, Y_i)$ for all i with $X_i \neq Y_i$.*

We call a member of $\mathcal{V}$ a vector of a valuated matroid and the conditions (VV1), (VV2), (VV3), (VVE) the vector axioms of a valuated matroid.

Example 2.2.7. Let us consider the following vectors in $(\mathbb{R} \cup \{\infty\})^5$:

$$\begin{aligned}
C_0 &= (\infty, \infty, \infty, \infty, 0),\\
C_1 &= (0, 1, \infty, \infty, \infty),\\
C_2 &= (0, \infty, 2, \infty, \infty),\\
C_3 &= (\infty, 1, 2, \infty, \infty).
\end{aligned}$$

Let us define

$$\mathcal{X} = \{C + k\mathbf{1} \mid C \in \{C_0, \ldots, C_3\}, k \in \mathbb{R}\}.$$

Then $\mathcal{X}$ can be shown to be a family of valuated circuits of some matroid. Let us call this matroid M_v. By (VC3) we can add a constant vector to any of these valuated circuits and we still have a circuit. For example

$$(1, 1, 1, 1, 1) + C_1 = (1, 2, \infty, \infty, \infty)$$

is also a circuit. We have that the first entry in C_1 and C_2 are equal and the second entry in C_1 is smaller than the corresponding entry in C_2. So we can apply (VCE) to C_1 and C_2 and we expect that a circuit of the form

$$(\infty, 1, \beta, \infty, \infty)$$

is in M_v, where $\beta \geq 2$. Our circuit C_3 satisfies this condition. We could have applied (VCE) to C_1 and C_2 differently, since we also have that the third entry of C_2 is smaller than the corresponding entry in C_1. So we expect a circuit of the form

$$(\infty, \gamma, 2, \infty, \infty)$$

in M_v, where $\gamma \geq 1$. Again, C_3 satisfies this condition.

The relation between vectors and valuated circuits is the following. For $\mathcal{S} \subseteq (\mathbb{R} \cup \{\infty\})^V$ let us define

$$\mathrm{Minsupp}(\mathcal{S}) = \{S \in \mathcal{S} \mid S \text{ has a nonempty minimal support in } \mathcal{S}\}.$$

Given a family $\mathcal{V}$ of vectors satisfying the vector axioms, $\mathcal{X} = \mathrm{Minsupp}(\mathcal{V})$ is the corresponding family of valuated circuits. On the other hand if $\mathcal{X} \subseteq (\mathbb{R} \cup \{\infty\})^V$ is the family of valuated circuits, $\mathcal{V} = \{X \mid X = \min(X^1, \ldots, X^k), k \geq 0, X^i \in \mathcal{X}\}$ is the family of vectors of the same valuated matroid. A proof of this fact can be found in [10, Theorem 3.6]. In particular, every valuated circuit is a valuated vector. Note also that if X and Y are two valuated circuits such that $\underline{X} = \underline{Y}$ then we must have $X = Y + \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$; see [10, Theorem 3.3].

One can see that the above definition of a matroid in terms of valuated circuits satisfies in particular properties (MC1)-(MC3) if we identify $\underline{\mathcal{X}} := \{\underline{X} \mid X \in \mathcal{X}\}$ with $\mathcal{C}$. Note that $\underline{\mathcal{V}} := \{\underline{V} \mid V \in \mathcal{V}\}$ are unions of circuits in $\mathcal{C}$. So a valuated matroid is a usual matroid with some extra structure added to it and, in particular, every valuated matroid is a matroid in the sense of Definition 2.2.1. Note that the valuated matroid from Example 2.2.7 has the same nonvaluated circuits as the matroid M we looked at when we defined usual matroids. The circuits C_0, C_1, C_2, C_3 correspond to nonvaluated circuits $\{5\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, respectively.

If M is a matroid or a valuated matroid on V we can define a certain substructure of it.

Definition 2.2.8. *Let M be a valuated matroid on the set V and let $V' \subset V$. Then the* restriction *of M to V' is the valuated matroid on the set V' whose circuits are the circuits of M contained in V'.*

Before finishing this section let us have a look at another example.

Example 2.2.9. Let us consider a valuated matroid M on the three element set $\{1,2,3\}$. We will describe it using valuated circuits in $\mathcal{X} \subset (\mathbb{R} \cup \{\infty\})^3$. Assume $X = (0,1,\infty)$ and $Y = (0,\infty,2)$ belong to $\mathcal{X}$ together with all the elements implied by the axiom (VC3). Axiom (VCE) implies, if we set $u = 1$ and $v = 2$, that $Z^1 = (\infty,1,x) \in \mathcal{X}$ for some $x \geq 2$. If we set $u = 1$ and $v = 3$ we get that $Z^2 = (\infty,y,2) \in \mathcal{X}$ for some $y \geq 1$. Consider first the case that $y > 1$ (note that $y \neq \infty$ - otherwise Y would not be a circuit). We have that $Z^1 + (y-1)\mathbf{1} = (\infty,y,x+y-1) \in \mathcal{X}$, where $x+y-1 > 2$. Using axiom (VCE) on Z^1 and Z^2, where $u = 2$ and $v = 3$, gives that $(\infty,\infty,2) \in \mathcal{X}$. But this would imply that Y is not a circuit. So $y = 1$ and similarly we must have $x = 2$. If we add to the set $\mathcal{X}$ all the elements implied by (VC3) then $\mathcal{X}$ is a set of valuated circuits. Later, using tropical ideals, we will see that it is easy to justify that $\mathcal{X}$ is indeed the family of valuated circuits, since $\mathcal{X}$ is the tropicalization of the degree one part of the ideal $\langle x+2y, x+4z\rangle \subset \mathbb{Q}[x,y,z]$ if we choose the 2-adic valuation.

2.3 Tropical semiring

In this section we will introduce the tropical semiring and we will see how to pass from classical arithmetic to the tropical arithmetic.

Definition 2.3.1. *The* tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ *is a semiring consisting of the set of real numbers with added symbol ∞ and the operations of addition and multiplication defined as*

$$x \oplus y := \min(x,y) \qquad \text{and} \qquad x \odot y := x + y,$$

where for any $x \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $x \oplus \infty = x$ and $x \odot \infty = \infty$.

The identity element for tropical addition is ∞ and 0 is the identity element for tropical multiplication. In the tropical semiring no elements, besides ∞, have an additive inverse: $\min(x, y) = \infty$ only if $x = y = \infty$. It follows that there is no subtraction. For example, the equation $2 \oplus x = 3$ has no solution over a tropical semiring. However, both addition and multiplication is commutative and associative. The distributive law also holds where $\odot$ takes precedence over $\oplus$.

Example 2.3.2. In the tropical semiring we have

$$2 \oplus 3 = 2, \qquad 2 \odot 4 = 6, \qquad (2 \oplus 3) \odot 4 = 2 \odot 4 = 6.$$

Let K be an arbitrary field. From now on we will denote the set $K \setminus \{0\}$ by K^*.

Definition 2.3.3. *Let K be a field. A* valuation *on K is a function* $\mathrm{val} : K \to \mathbb{R} \cup \{\infty\}$ *satisfying the following three axioms:*

1. $\mathrm{val}(a) = \infty$ *if and only if* $a = 0$,
2. $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$,
3. $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ *for all* $a, b \in K^*$.

The image of K^* under the valuation map, denoted Γ_{val}, is an additive subgroup of the real numbers $\mathbb{R}$ which is called the value group. *A valuation is called* trivial *if* $\mathrm{val}(a) = 0$ *for all $a \in K^*$. Otherwise it is called* non-trivial.

Note that

$$\mathrm{val}(1) = \mathrm{val}(1 \cdot 1) = \mathrm{val}(1) + \mathrm{val}(1),$$

so $\mathrm{val}(1) = 0$. Then it follows

$$0 = \mathrm{val}(1) = \mathrm{val}((-1) \cdot (-1)) = \mathrm{val}(-1) + \mathrm{val}(-1),$$

so $\mathrm{val}(-1) = 0$.

Example 2.3.4. Take K to be the field of rational numbers and let p be an arbitrary prime number. Any $x \in \mathbb{Q}$ can be written in a unique way as $x = p^k \frac{a}{b}$, where a, b, k are integers and p does not divide a or b. Let $\mathrm{val}_p : \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ be such that $x = p^k \frac{a}{b} \mapsto k$. We call this valuation a p-adic valuation on the field $\mathbb{Q}$. For example, $\mathrm{val}_3(3) = 1$, $\mathrm{val}_3(\frac{4}{3}) = -1$.

Example 2.3.5. The field of Puiseux series $\mathbb{C}\{\{t\}\}$ with coefficients in $\mathbb{C}$ consists of formal power series of the form

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \cdots ,$$

where c_i are non-zero complex numbers and $a_1 < a_2 < a_3 < \cdots$ are rational numbers that have a common denominator. The natural valuation for this field is $\mathrm{val} : \mathbb{C}\{\{t\}\} \to \mathbb{R} \cup \{\infty\}$ which takes $c(t)$ to the lowest exponent a_1 in the series expansion of $c(t)$. For example,

$$\mathrm{val}(2t^{-\frac{2}{3}} + 4t + t^4) = -\frac{2}{3}, \qquad \mathrm{val}(2) = 0.$$

We can now define the tropicalization of a polynomial in the polynomial ring $S = K[x_1, \ldots, x_n]$. Informally speaking, when we tropicalize a polynomial we replace normal addition with tropical addition and standard multiplication with tropical multiplication. We also replace coefficients with their image under a valuation.

Definition 2.3.6. *Let $f(x) = \sum_{\mathbf{u} \in \mathbb{N}^{n+1}} c_{\mathbf{u}} x^{\mathbf{u}}$ be an element of S. The* tropicalization *of $f(x)$ is defined to be*

$$\mathrm{trop}(f(x)) := \bigoplus_{\mathbf{u} \in \mathbb{N}^{n+1}} \mathrm{val}(c_{\mathbf{u}}) \odot x^{\mathbf{u}}.$$

A *tropical monomial* in variables $x_1, \ldots, x_n$ is any tropical product of these variables. Tropicalization of a polynomial gives us a *tropical polynomial*, i.e., a finite linear combination of tropical monomials $a_0 \odot m_0 \oplus a_1 \odot m_1 \cdots \oplus a_k \odot m_k$, where a_i are elements of $\bar{\mathbb{R}}$ and m_i are tropical monomials. Tropical polynomial semiring is a set of all tropical polynomials which is equipped with tropical addition and tropical multiplication.

Example 2.3.7. Consider a polynomial $f(x) = x^2+2x+1$. Replacing multiplication and addition with their tropical versions and using the 2-adic valuation we get

$$\mathrm{trop}(f)(x) = 0 \odot x \odot x \oplus 1 \odot x \oplus 0 = \min(2x, x+1, 0).$$

For a polynomial $g(x, y) = (t^3 + 5t^4)x + \frac{1}{2}y + 3t$ with coefficients in $\mathbb{C}\{\{t\}\}$ we get

$$\mathrm{trop}(g)(x, y) = \min(x + 3, y, 1).$$

The tropicalization of an ideal I is an ideal in the tropical polynomial semiring which is generated by the tropicalization of all the polynomials in I.

Definition 2.3.8. *Let I be an ideal in $K[x_1, \ldots, x_n]$ and let us fix a valuation on K. Then the* tropicalization *of I is defined as* $\mathrm{trop}(I) = \langle \mathrm{trop}(f) \mid f \in I \rangle \subset \bar{\mathbb{R}}[x_1 \ldots, x_n]$.

In the next section we will see that the tropicalization of any ideal gives a tropical ideal.

2.4 Tropical ideals

We have seen how to tropicalize an ideal in $K[x_1, \ldots, x_n]$. In this section we will define tropical ideals and study some of their properties.

Tropical ideals are a subset of ideals in the tropical polynomial semiring.

Definition 2.4.1. *A homogeneous ideal $I \subset \bar{\mathbb{R}}[x_0, \ldots, x_n]$, where $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, is a* tropical ideal *if the set of coefficient vectors of polynomials of degree d in I forms the set of vectors of a valuated matroid.*

Before unpacking this definition we need to define several notions.

Definition 2.4.2. *Let $p = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ be a classical or a tropical homogeneous polynomial. The* support *of p is the subset of $\mathbb{N}^n$ defined by* $\mathrm{supp}(p) := \{\mathbf{u} \mid c_{\mathbf{u}} \neq 0\}$ *if $p \in K[x_1, \ldots, x_n]$ or by* $\mathrm{supp}(p) := \{\mathbf{u} \mid c_{\mathbf{u}} \neq \infty\}$ *if $p \in \bar{\mathbb{R}}[x_1, \ldots, x_n]$.*

If $p \in I$, where I is a homogeneous ideal/tropical ideal, we say that p is of minimal support if there is no polynomial p' in I such that $\mathrm{supp}(p') \subsetneq \mathrm{supp}(p)$.

When it is clear from the context what we mean, we will often abuse notation and give as the support of a polynomial the set of its monomials whose coefficients are non-zero/not ∞. So even though according to our definition we have that $\mathrm{supp}(x^2 + xy) = \{(2, 0), (1, 1)\}$ we will often say that support of $x^2 + xy \in K[x, y]$ is the set $\{x^2, xy\}$.

Example 2.4.3. In the ideal $I = \langle x + y \rangle \subset \mathbb{C}[x, y]$ the polynomial $x^2 + xy$ is of minimal support whereas $(x^2 + xy) + (xy + y^2) = x^2 + 2xy + y^2$ is not, since $\mathrm{supp}(x^2 + xy) = \{x^2, xy\} \subset \{x^2, xy, y^2\} = \mathrm{supp}(x^2 + 2xy + y^2)$.

It follows from Definitions 2.4.1 and 2.4.2 that in a tropical ideal, homogeneous polynomials of minimal support correspond to the valuated circuits of the underlying matroid and if a given set of monomials of the same degree is not a support of any polynomial in the ideal, and neither is any of its subsets, then this set is independent in the underlying matroid. We also have that any homogeneous polynomial in a tropical ideal is a tropical sum of valuated circuits. This last property follows from the correspondence between the set of valuated circuits and valuated vectors in a valuated matroid as explained in Section 2.2.

So we have the following definition.

Definition 2.4.4. *Let I be a tropical ideal in $\bar{\mathbb{R}}[x_1, \ldots, x_n]$. A set $\{\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r}\}$ of monomials of degree d in $\bar{\mathbb{R}}[x_, \ldots, x_n]$ is called a* circuit *if there exists a polynomial of the form $\sum_{i=1}^{r} \alpha_i \mathbf{x}^{\mathbf{u_i}}$ in I, where $\alpha_i \in \mathbb{R}$, and if this set is inclusion minimal with respect to this property. A set $\{\mathbf{x}^{\mathbf{u}_1}, \ldots, \mathbf{x}^{\mathbf{u}_r}\}$ is said to be* independent *if there is no polynomial of the form $\sum_{i=1}^{r} \alpha_i \mathbf{x}^{\mathbf{u}_i}$ in I (we allow all but one of α_i to be ∞ in the last equation). Polynomials in I which are not of minimal support are called* vectors.

Recall that we discussed before that every valuated matroid has an underlying ordinary matroid. This property carries over to tropical ideals. Namely, if I is a tropical ideal in $\bar{\mathbb{R}}[x_1, \ldots, x_n]$ then each homogeneous degree has the underlying structure of an ordinary matroid. In particular, if all polynomials of a tropical ideal have coefficients in the set $\{0, \infty\} \subset \bar{\mathbb{R}}$ then we write that $I \subset \mathbb{B}[x_1, \ldots, x_n]$, where $\mathbb{B}$ is the Boolean subsemiring of $\bar{\mathbb{R}}$: $\mathbb{B} := (\{0, \infty\}, \oplus, \odot)$.

Before studying general properties of the tropical ideals let us look at an example.

Example 2.4.5. We will construct a tropical ideal J in $\mathbb{B}[x, y]$. First, let us note that what we need to do is to give a sequence of matroids M_d, where $d \geq 1$, such that for a given d the matroid M_d is on the set of monomials $\{x^d, x^{d-1}y, \ldots, y^d\}$. The matroids must satisfy a condition that if C is a circuit in M_d, then both xC and yC must be circuits in M_{d+1}. Since we look at an ideal in $\mathbb{B}[x, y]$, all the coefficients that are not ∞ must be equal to 0. We will denote by J_d the homogeneous degree d part of J.

Assume that J_1 consists of exactly one polynomial $x \oplus y$ (this is a shorthand notation for $0 \odot x \oplus 0 \odot y$). This means that the underlying matroid M_1 on a two element set $\{x, y\}$ has a unique circuit $\{x, y\}$. The degree two part of J must contain polynomials $x \odot (x \oplus y) = x^2 \oplus xy$ and $y \odot (x \oplus y) = xy \oplus y^2$. So a matroid on a three element monomial set $\{x^2, xy, y^2\}$ must have the following two vectors: $\{x^2, xy\}$ and $\{xy, y^2\}$. Assume that we want these vectors to be circuits. Then by the circuit elimination axiom we get that $\{x^2, y^2\}$ must also be a circuit. So it follows that a polynomial $x^2 \oplus y^2$ must belong to J_2. Continuing this way, we can construct the ideal J with the property that for each positive $d \in \mathbb{N}$, J_d is the set of all polynomials in x, y of degree d and support of size two. We will see the details of this in later chapters. In terms of matroids, for each d, M_d is the uniform matroid of rank one on the set $\{x^d, x^{d-1}y, \ldots, y^d\}$.

Note that the ideal constructed in the above example is equal to $\mathrm{trop}(I)$, where I is an ideal in $\mathbb{C}[x, y]$ defined by $I = \langle x+ay \rangle$, where a is an arbitrary element of $\mathbb{C}^*$ and we use the trivial valuation for tropicalization. In general, whenever for a tropical ideal $J \subset \bar{\mathbb{R}}[x_1, \ldots, x_n]$ there exists a valuated field K with a valuation v, and an ideal $I \subset K[x_1, \ldots, x_n]$ such that $\mathrm{trop}(I) = J$ with respect to v, then we say that the ideal J is *realizable*. It is true in general that tropicalization of a classical ideal is a tropical ideal; a proof of this fact can be found in [6, Example 2.2]. Not every tropical ideal is realizable though, see [6, Example 2.8] for an example.

We use the definition of Hilbert function for tropical ideals as introduced in [6].

Definition 2.4.6. *Let $S = \bar{\mathbb{R}}[x_1, \ldots, x_n]$ and let $I \subset S$ be a tropical ideal. Then the* Hilbert function *of I is defined to be the rank of the underlying valuated matroid in each degree, i.e.,*

$$H_{S/I}(d) = \mathrm{rank}(\mathrm{Mat}(I_d)),$$

where $\mathrm{Mat}(I_d)$ *is the valuated matroid of I_d.*

With this definition it turns out that Hilbert function is preserved under tropicalization. This fact was first proved in [4]. The proof included below can be found in [6, page 12].

Proposition 2.4.7. *Let I be an ideal in $S = K[x_1, \ldots, x_n]$ with Hilbert function $H_{S/I}$. Then the tropical ideal* $\mathrm{trop}(I)$ *has the same Hilbert function as I.*

Proof. Consider the degree d part of I. As a vector space it is generated by $\binom{d+n-1}{d} - H_{S/I}(d)$ polynomials. We can represent it as a $\left(\binom{d+n-1}{d} - H_{S/I}(d)\right) \times \binom{d+n-1}{d}$ matrix, whose columns are labelled by all monomials of degree d in S. The orthogonal complement of the row space of this matrix is a $H_{S/I}(d) \times \binom{d+n-1}{d}$ matrix M', such that a $H_{S/I}(d) \times H_{S/I}(d)$ minor of M' is zero if and only if a corresponding set of monomials forms a dependent set in the degree d part of $\mathrm{trop}(I)$. In particular, the rank of matrix M', which is $H_{S/I}(d)$, is the rank of the matroid of $\mathrm{trop}(I)_d$, from which the claim follows. $\square$

As an example, remember that we have seen in Example 2.4.5 that each homogeneous degree of the ideal we constructed had the underlying structure of a matroid of rank one. We also claimed that this ideal is a tropicalization of a point ideal, which also has Hilbert function one.

Definition 2.4.8. *Let* $I \subset K[x_1, \ldots, x_n]$ *be an ideal and fix* $f \in K[x_1, \ldots, x_n]$*. Then the* saturation *of* I *with respect to* f *is the set* $(I : f^\infty) := \{g \in K[x_1, \ldots, x_n] : f^m g \in I$ *for some* $m > 0\}$*.*

We can replace K with $\bar{\mathbb{R}}$ in the above to get exactly the same definition for the tropical ideals. It can be shown that $(I : f^\infty)$ is an ideal in $K[x_1, \ldots, x_n]$. For details see, for example, [1].

Example 2.4.9. Let $J \subset \bar{\mathbb{R}}[x_1, \ldots, x_n]$ be a tropical ideal. If J is saturated with respect to the product of its variables it means that $J : (x_1 \cdots x_n)^\infty = J$. In other words, whenever for some monomials $m, m_1, \ldots, m_k$ we have that a polynomial of the form $m(m_1 \oplus \cdots \oplus m_k)$ is in J, it implies that $m_1 \oplus \cdots \oplus m_k$ is also in J.

Similarly as in the classical case, we can define tropical elimination ideals. Remember that in the classical case we have the following.

Definition 2.4.10. *Given the ideal* $I = \langle f_1, \ldots, f_s \rangle \subset K[x_1, \ldots, x_n]$ *the* l*-th* elimination ideal I_l *is the ideal of* $K[x_{l+1}, \ldots, x_n]$ *defined by* $I_l = I \cap K[x_{l+1}, \ldots, x_n]$*.*

Before defining tropical elimination ideals we must make sure that the intersection of a tropical ideal $J \subset \bar{\mathbb{R}}[x_1, \ldots, x_n]$ with a smaller polynomial ring will result in a tropical ideal.

Lemma 2.4.11. *Let* J *be a tropical ideal in* $\bar{\mathbb{R}}[x_1, \ldots, x_n]$*. Then* $J \cap \bar{\mathbb{R}}[x_{l+1}, \ldots, x_n] \subset \bar{\mathbb{R}}[x_{l+1}, \ldots, x_n]$ *is also a tropical ideal.*

Proof. Let $J' = J \cap \bar{\mathbb{R}}[x_{l+1}, \ldots, x_n] \subset \bar{\mathbb{R}}[x_{l+1}, \ldots, x_n]$. The only non-trivial thing to prove is that for each degree d, J' has the underlying structure of a valuated matroid. But J'_d is a matroid, which is the restriction of the matroid of J_d to monomials not containing any of $x_1, \ldots, x_l$, so we are done. $\square$

Definition 2.4.12. *We call the ideal* J_l *defined in 2.4.11 the* l*-th* elimination tropical ideal.

We are ready to prove that tropicalization commutes with elimination. We will need this result in later work when proving realizability.

Lemma 2.4.13. *Let* I *be an ideal in* $K[x_1, \ldots, x_n]$*. Then for any* $0 \leq l < n$ *we have* $\mathrm{trop}(I \cap K[x_{l+1}, \ldots, x_n]) = \mathrm{trop}(I) \cap \bar{R}[x_{l+1}, \ldots, x_n]$*.*

Proof. Note that it is enough to consider only the polynomials of minimal support. First we will show $\subseteq$ inclusion. Assume p is a non-zero polynomial of minimal support in $I \cap K[x_{l+1}, \ldots, x_n]$. Then $p \in I$ and so $\mathrm{trop}(p) \in \mathrm{trop}(I)$. Since p is a polynomial in in $K[x_{l+1}, \ldots, x_n]$, $\mathrm{trop}(p) \in \bar{R}[x_{l+1}, \ldots, x_n]$ and we are done.

For $\supseteq$ inclusion, note that if p is a polynomial of minimal support in $\mathrm{trop}(I) \cap \bar{R}[x_{l+1}, \ldots, x_n]$ then there exists $p' \in I$ such that $\mathrm{trop}(p') = kp$ for some constant k and $p' \in K[x_{l+1}, \ldots, x_n]$. So p' is such that $p' \in I \cap K[x_{l+1}, \ldots, x_n]$ and $\mathrm{trop}(p') = kp$. This implies $p \in \mathrm{trop}(I \cap K[x_{l+1}, \ldots, x_n])$ and we are done. $\square$

We finish this section by giving the conventions we will use throughout the rest of this work. In what follows, unless stated otherwise, by an ideal (respectively tropical ideal) we mean a homogeneous ideal (respectively tropical ideal) which is saturated with respect to the product of its variables. Whenever we say tropical ideals with (respectively without) coefficients we mean that the coefficients are in the set $\bar{\mathbb{R}}$ (respectively $\mathbb{B}$) and we consider the underlying valuated (respectively classical) matroids. We denote by $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr,k,}\mathbb{B}}$ the set of all tropical ideals in n variables with Hilbert function k and with polynomials whose coefficients are in the set $\mathbb{B}$. Similarly, $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr,k,}\bar{\mathbb{R}}}$ denotes the set of all tropical ideal in n variables with Hilbert function k and with polynomials whose coefficients are in the set $\bar{\mathbb{R}}$. If $n = 2$ or $n = 3$, we replace n with x, y or x, y, z, respectively. If a tropical ideal $J \subset \bar{\mathbb{R}}[x_1, \ldots, x_n]$ is realizable over a polynomial ring $K[x_1, \ldots, x_n]$ we say that the tropical ideal J is realizable over a field K. If K is a field, K^* denotes $K \setminus \{0\}$.

Chapter 3

Hilbert function one

Let $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$ be the set of all tropical ideals (saturated and not saturated) with Hilbert function one in $\bar{\mathbb{R}}[x_1, \ldots, x_n]$. It was proven by Maclagan and Rincón in [6] that every tropical ideal in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$ comes from a tropicalization of a point ideal in $K[x_1, \ldots, x_n]$, where K is a field with a non-trivial valuation. It is assumed that all tropical coefficients in the ideals considered belong to the value group of K, Γ_{val}. In this chapter we discuss this statement and the proof.

Let us first see the possibilities for the degree one of an ideal $J \in \mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$. As discussed in Section 2.4, the set J_1 must have the underlying structure of a valuated matroid of rank one on the set $\{x_1, \ldots, x_n\}$. In such matroids, since the bases are a set of singletons, the circuits partition the ground set into a collection of singletons and pairs.

We claim that after choosing the degree one part of $J \in \mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$, we have no further choices for the higher degrees of J.

Proposition 3.0.1. *Any tropical ideal in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,\textit{non-sat}}}$ is determined by its degree one part.*

Proof. Let us assume that any pair of distinct variables x_i, x_j forms a valuated circuit. First we will prove that it is not possible that a monomial $\mathbf{x}^{\mathbf{u}}$ of degree d is a circuit in J_d. For a contradiction, assume that $\mathbf{x}^{\mathbf{u}} \in J_d$, where $\mathbf{u} = (u_1, \ldots, u_n) \in N^d$ and for some $i \in \{1 \ldots n\}$, $u_i \neq 0$. For all $j \in \{1, \ldots, n\} \setminus \{i\}$ we have $x_i \oplus a_{ij} x_j \in J_1$ for some $a_{i,j} \in \Gamma_{\text{val}}$. It follows that $\mathbf{x}^{\mathbf{u}} \oplus a_{ij} \mathbf{x}^{\mathbf{u}-\mathbf{e}_i} x_j \in J_d$, where $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{N}^n$. By the valuated circuit elimination axiom it follows that $\mathbf{x}^{\mathbf{u}-\mathbf{e}_i} x_j \in J_d$. Since j was arbitrary it means that if $\mathbf{x}^{\mathbf{u}} \in J_d$ then all monomials of degree d are in J_d. This contradicts J having Hilbert function one.
Since the biggest circuits in J are of support strictly less than three, what is left to show is that for any two monomials $\mathbf{x}^{\mathbf{u}}$, $\mathbf{x}^{\mathbf{v}}$ we can determine the coefficient c in

$\mathbf{x}^{\mathbf{u}} \oplus c\mathbf{x}^{\mathbf{v}}$ using only the degree one part of J. Let i_0 and j_0 be the first positive and the first negative entry in $\mathbf{u} - \mathbf{v}$, respectively. We know the coefficient $c_{i_0j_0}$ in $x_{i_0} \oplus c_{i_0j_0}x_{j_0}$ so we also know that the polynomial

$$\mathbf{x}^{\mathbf{u}'}(x_{i_0} \oplus c_{i_0j_0}x_{j_0}) = \mathbf{x}^{\mathbf{u}} \oplus c_{i_0j_0}\mathbf{x}^{\mathbf{u}_1} \tag{3.1}$$

is in J, where $\mathbf{u}' = \mathbf{u} - \mathbf{e}_{i_0}$ and $\mathbf{u}_1 = \mathbf{u} - \mathbf{e}_{i_0} + \mathbf{e}_{j_0}$. Continuing in the same way, let i_1 and j_1 be the first positive and the first negative entry in $\mathbf{u}_1 - \mathbf{v}$, respectively. From the equation $x_{i_1} \oplus c_{i_1j_1}x_{j_1}$ we know the coefficient $c_{i_1j_1}$ in

$$\mathbf{x}^{\mathbf{u}''}(x_{i_1} \oplus c_{i_1j_1}x_{j_1}) = \mathbf{x}^{\mathbf{u}_1} \oplus c_{i_1j_1}\mathbf{x}^{\mathbf{u}_2}, \tag{3.2}$$

where $\mathbf{u}'' = \mathbf{u}_1 - \mathbf{e}_{i_1}$ and $\mathbf{u}_2 = \mathbf{u}_1 - \mathbf{e}_{i_1} + \mathbf{e}_{j_1}$. After tropically multiplying the polynomial (3.2) by $c_{i_0j_0}$, we can use the valuated circuit elimination axiom on

$$\begin{gathered} \mathbf{x}^{\mathbf{u}} \oplus c_{i_0j_0}\mathbf{x}^{\mathbf{u}_1} \quad \text{and} \\ c_{i_0j_0}\mathbf{x}^{\mathbf{u}_1} \oplus (c_{i_0j_0} \odot c_{i_1j_1})\mathbf{x}^{\mathbf{u}_2} \end{gathered}$$

to obtain the coefficient $c_2 = c_{i_0j_0} \odot c_{i_1j_1}$ in $\mathbf{x}^{\mathbf{u}} \oplus c_2\mathbf{x}^{\mathbf{u}_2}$. If we repeat the above steps until we obtain $\mathbf{u}_k = \mathbf{v}$, for some integer k, we will get the coefficient we are looking for.

The case when J_1 contains a mixture of singletons and pairs follows immediately from the previous case, after noticing that a monomial in degree d constitutes a circuit on its own if and only if it contains at least one of the variables corresponding to one of the singletons in J_1. □

Note that even though the above proposition states that a tropical ideal in $\mathcal{I}^{\mathrm{tr},1,\bar{\mathbb{R}}}_{\mathrm{n,non\text{-}sat}}$ is determined by its degree one part, we do not know a priori that all the possibilities for the degree one considered above belong to some tropical ideal in $\mathcal{I}^{\mathrm{tr},1,\bar{\mathbb{R}}}_{\mathrm{n,non\text{-}sat}}$. The existence of all such tropical ideals will be a consequence of Theorem 3.0.2.

Recall that tropicalization of any classical ideal gives a tropical ideal and it preserves its Hilbert function (Proposition 2.4.7). So to prove that all tropical ideals in $\mathcal{I}^{\mathrm{tr},1,\bar{\mathbb{R}}}_{\mathrm{n,non\text{-}sat}}$ are realizable it is enough to show that for any tropical ideal $J \in \mathcal{I}^{\mathrm{tr},1,\bar{\mathbb{R}}}_{\mathrm{n,non\text{-}sat}}$ there exists a classical ideal I with Hilbert function one such that $J_1 = (\mathrm{trop}(I))_1$.

Theorem 3.0.2. *Every tropical ideal in $\mathcal{I}^{\mathrm{tr},1,\bar{\mathbb{R}}}_{\mathrm{n},\mathit{non\text{-}sat}}$ is realizable.*

Proof. From Proposition 3.0.1 we have an upper bound for all possible tropical ideals in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$. We will show that all potential degree one parts of ideals in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$ are tropicalizations of degree one part of ideals with Hilbert function one in $K[x_1,\ldots,x_n]$.

Let J be an ideal in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$. In the case where J_1 is generated only by variables, say $x_2,\ldots,x_n$, we have that $I=\langle x_2,\ldots,x_n\rangle\subset K[x_1,\ldots,x_n]$ is its realization.

Note that if an ideal $J\in\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}\subset\bar{\mathbb{R}}[x_1,\ldots,x_n]$ is such that for some $l\geq 2$, $x_{l+1},x_{l+2},\ldots,x_n$ are circuits in J_1 and they are the only circuits of size one, then in order to describe the ideal J it is enough to give coefficients in the binomials $x_1\oplus c_{12}x_2$, $x_1\oplus c_{13}x_3,\ldots,x_1\oplus c_{1l}x_l$. Indeed, the coefficient c_{ij} in a binomial $x_i\oplus c_{ij}x_j$, where $1<i,j\leq l$, can be determined by applying the valuated circuit elimination axiom to polynomials

$$\begin{aligned}&x_1\oplus c_{1i}x_i \quad \text{and}\\&x_1\oplus c_{1j}x_j\end{aligned}$$

and tropically multiplying the result by $-c_{1i}$. This way we get $c_{ij}=c_{1j}-c_{1i}$. By Proposition 3.0.1 we can now determine all the polynomials in J.

So if $x_{l+1},\ldots,x_n$ are the only one element circuits in J_1, let us consider the ideal $I=\langle x_1-t^{c_{12}}x_2,\ldots,x_1-t^{c_{1l}}x_l,x_{l+1},\ldots,x_n\rangle\subset K[x_1\ldots,x_n]$, where t^c denotes the preimage of c under the valuation map. We have $J_1=(\text{trop}(I))_1$. Note also that I is a point ideal so it has Hilbert function one and we are done. $\square$

Proposition 3.0.1 together with Theorem 3.0.2 allows us to describe completely all tropical ideals in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$.

Corollary 3.0.3. *Up to symmetry, all possible degree one parts of tropical ideals in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,}non\text{-}sat}$ are generated by the sets of the form $\{x_1\oplus c_{1i}x_i \mid i\in T\}\cup\{x_j\mid j\in\{2,\ldots,n\}\setminus T\}$, where $T=\{2,\ldots,k\}$, $1\leq k\leq n$ and $c_{1i}\in\Gamma_{\text{val}}$. Each of these sets can be uniquely expanded to a tropical ideal in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,}non\text{-}sat}$. Note that for $k=1$ we have $T=\emptyset$.*

Proof. In the proof of Proposition 3.0.1 we saw that, up to symmetry, there is at most one tropical ideal in $\mathcal{I}^{\text{tr},1,\bar{\mathbb{R}}}_{\text{n,non-sat}}$ having the above set in its degree one part. In the proof of Theorem 3.0.2 we showed that any set of tropical polynomials of this form is a tropicalization of the degree one part of some point ideal $I\in K[x_1,\ldots,x_n]$. Since the tropicalization of every classical ideal is a tropical ideal, we are done. $\square$

Chapter 4

Hilbert function two without coefficients

In this chapter we will give a description of saturated tropical ideals with Hilbert function two in $\mathbb{B}[x_0, \ldots, x_n]$, i.e., ideals in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$. We will see that as opposed to classical ideals with Hilbert function two, tropical ideals are not determined by their degree two part.

Let J be a tropical ideal in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$. Then for any d, J_d has the underlying structure of a matroid of rank two. In such matroids the biggest independent sets have support of size two. Moreover, note that if a matroid has a circuit of size one, it means that the corresponding ideal has a monomial. Since we look at saturated ideals, this cannot be the case. So we know that the underlying matroids do not have circuits of size one. To describe such matroids it is enough if for each pair of elements of the ground set we say whether they are dependent or independent. If they are dependent it follows they must form a circuit.

4.1 Tropical ideals without coefficients in three variables

As a warm-up, we will look at tropical ideals in three variables. Let us consider possibilities for the degree one part of a tropical ideal $J \in \mathcal{I}_{\mathrm{x,y,z}}^{\mathrm{tr},2,\mathbb{B}}$, where $J \subset \mathbb{B}[x, y, x]$. We have that J_1 cannot be empty as the rank of the underlying matroid would be three. Since the ideal is saturated, it cannot have any monomials. One possibility for J_1 is that it has exactly one polynomial, and it is of the form $x \oplus y \oplus z$. For another choice for J_1, assume that J_1 has a binomial. Without loss of generality, let the binomial be $x \oplus z$. Assume there is another binomial in J_1 and

	x^2, y^2	x^2, yz	x^2, z^2	xy, z^2	y^2, xz	y^2, z^2
1	1	0	1	0	0	1
2	1	0	1	0	1	1
3	1	0	1	1	0	1
4	1	0	1	1	1	0
5	1	0	1	1	1	1
6	1	1	0	1	0	1
7	0	1	0	1	1	0
8	1	1	0	1	1	1
9	1	1	1	0	0	1
10	1	1	1	0	1	1
11	0	1	1	0	1	1
12	1	1	1	1	0	1
13	1	1	1	1	1	0
14	0	1	1	1	1	1
15	1	1	1	1	1	1

Table 4.1: Candidates for the underlying matroid of the degree two part of a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$. By 0, respectively 1, we denote that the corresponding pair of monomials forms a dependent, respectively independent, set. Note that not all of the above possibilities extend to tropical ideals in degree 3.

without loss of generality let it be $x \oplus y$. Then, by the circuit elimination axiom, $y \oplus z$ is also in J_1 and it follows that the rank of the underlying matroid is one. So J_1 cannot contain more than one binomial. Summing up, without loss of generality we can assume that J_1 has exactly one polynomial and it is either $x \oplus z$ or $x \oplus y \oplus z$.

Let us consider the case when $p = x \oplus y \oplus z$ is in J_1. This implies that $x \odot p$, $y \odot p$ and $z \odot p$ are all in J_2 and they are also polynomials of minimal support. To see this, for a contradiction assume that at least one of these polynomials is not of minimal support. Without loss of generality assume that $x \odot p$ is not of minimal support. Since we do not allow one element circuits, we must have that a two element subset of $\{x^2, xy, xz\}$ corresponds to a polynomial of minimal support in J_2. Since J is saturated, this would imply that a two element subset of $\{x, y, z\}$ corresponds to a polynomial of minimal support in J_1. But this is a contradiction since p was of minimal support. Let us see what other circuits there are in J_2. Since J is saturated and none of $x \oplus y$, $x \oplus z$ or $y \oplus z$ is in J, the following pairs of monomials must be independent: $\{x^2, xy\}$, $\{x^2, xz\}$, $\{xy, y^2\}$, $\{xz, z^2\}$, $\{y^2, yz\}$ and $\{yz, z^2\}$. Considering possible cases for the dependency for the remaining pairs we get that there are 15 matroids which potentially can correspond to the degree two part of an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$.They are shown in Table 4.1.

It turns out that the cases 2, 3 and 9 from Table 4.1 cannot form the homogeneous degree two part of a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ because it cannot happen that exactly two of the polynomials $x^2 \oplus yz$, $y^2 \oplus xz$, $z^2 \oplus xy$ are the only polynomials of support two in J_2.

Lemma 4.1.1. *Let $J \subset \mathbb{B}[x, y, z]$ be a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ such that $J_1 = \{x \oplus y \oplus z\}$ and $x^2 \oplus yz$, $xz \oplus y^2 \in J_2$. Then $xy \oplus z^2$ is also in J_2.*

Proof. If $x^2 \oplus yz$ and $xz \oplus y^2$ are both in J then $x^2z \oplus yz^2$ and $x^2z \oplus xy^2$ are also in J. By the circuit elimination axiom we get $xy^2 \oplus yz^2 \in J$, which by saturation implies $xy \oplus z^2 \in J$. □

We will show later in this chapter that all the other cases can be expanded to tropical ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$, and for the cases 1, 4, 6, 7, 11 there is a unique way of doing so.

4.2 Description of tropical ideals with Hilbert function two.

In this section we will give the description of all tropical ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{n}}$, that is tropical ideals in $\mathbb{B}[x_1, \ldots, x_n]$ with Hilbert function two and saturated with respect to $x_1x_2\cdots x_n$.

We will start by giving some intuition behind the main theorem and the proof.
In order to describe an ideal J in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{n}}$, for each degree d we need to specify the corresponding matroid. This means that for each set of monomials of a given degree, we have to say whether they form a polynomial of minimal support in J or not. It turns out that it is enough to study only binomials in J. So we can describe an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{n}}$ through integer vectors in $\mathbb{Z}^n$, where, up to a sign, we identify a binomial

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \oplus x_1^{a_1}x_2^{b_2}\cdots x_n^{b_n}$$

with a vector

$$(a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

This map is not injective, but since we look at ideals saturated with respect to $x_1x_2\cdots x_n$ this is not a problem. Note that since we look at homogeneous binomials it means that the entries in each such vector add up to zero. Also, vectors $\mathbf{u}$ and $-\mathbf{u}$ correspond to the same binomial. So if L is the lattice of all integer points in $\mathbb{Z}^n$, such that if $(a_1, \ldots, a_n) \in L$ then $a_1 + \cdots + a_n = 0$, then a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{n}}$ can be identified with a sublattice of L.

Throughout this chapter, by L we denote the lattice consisting of all integer points in $\mathbb{Z}^n$ which lie in the hyperplane defined by $x_1 + x_2 + \cdots + x_n = 0$. By a proper sublattice of L we mean any sublattice $A \subset L$ such that $A \neq L$. We also allow an empty sublattice.

Let us consider the following construction which sends a tropical ideal into a sublattice of the lattice L. Let J be an arbitrary tropical ideal in $\mathbb{B}[x_1, \ldots, x_n]$. For every homogeneous binomial $p = \mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{v}}$ in J let us define the vector $\mathbf{a} = \mathbf{u} - \mathbf{v}$, which lies in the lattice L. With the introduced notation, we have the following definition.

Definition 4.2.1. *The lattice of an ideal $J \in \mathbb{B}[x_1, \ldots, x_n]$ is a sublattice of L generated by all homogeneous binomials in J.*

Now we will introduce a map which takes a sublattice of L and gives a tropical ideal.
For an element $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ let us define $|\mathbf{u}| := \sum_{i=1}^{n} u_i$. Let A be a sublattice of the lattice L. Let us consider pairs of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ such that $\mathbf{u} - \mathbf{v} \in A$. Notice that we have $|\mathbf{u}| = |\mathbf{v}|$. For each $d > 0$, let $\mathcal{C}_d$ be the set whose elements are all pairs of distinct monomials of degree d in $\mathbb{B}[x_1, \ldots, x_n]$ such that $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\} \in \mathcal{C}_d$ if $\mathbf{u} - \mathbf{v} \in A$, and all triples of monomials not having as a subset one of the above pairs. With this notation we have the following statement.

Lemma 4.2.2. *The set $\cup_1^{\infty} \mathcal{C}_d$ is the set of supports of all homogeneous polynomials of minimal support of an ideal $J \in \mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$.*

Proof. First we will check that for each d there exists a unique matroid M_d on the set of all monomials of degree d in $\mathbb{B}[x_1, \ldots, x_n]$, such that each $\mathcal{C}_d$ is the set of circuits of M_d. So let us show that these sets satisfy matroid axioms for circuits.

Axioms $(MC1)$ and $(MC2)$ follow immediately from the definition of the sets $\mathcal{C}_d$. In order to show that axiom (MCE) is satisfied, we will consider three cases depending on the sizes of the circuits considered.

Assume first that for some d, $C_1 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_1}\}$ and $C_2 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_2}\}$, where all of $\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2$ are different, are in $\mathcal{C}_d$. It follows $\mathbf{u} - \mathbf{v}_1$ and $\mathbf{u} - \mathbf{v}_2$ are in A and so $\mathbf{v}_1 - \mathbf{v}_2$ is also in A. But this means that $\{\mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{v}_2}\}$ is in $\mathcal{C}_d$ and we are done.

Assume now that $C_1 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{w}_1}\}$ and $C_2 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_2}\}$, where all of $\mathbf{u}, \mathbf{v}_1, \mathbf{w}_1, \mathbf{v}_2$ are different, are in $\mathcal{C}_d$. Notice that we cannot have that $\mathbf{v}_1 - \mathbf{v}_2$ is in A, as since $\mathbf{u} - \mathbf{v}_2$ is in A, this would imply that $\mathbf{u} - \mathbf{v}_1$ is in A as well. But then C_1 would not be in $\mathcal{C}_d$. Similarly it cannot happen that $\mathbf{w}_1 - \mathbf{v}_2$ is in A. So we have that $\{\mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{w}_1}, \mathbf{x}^{\mathbf{v}_2}\}$ is in $\mathcal{C}_d$ and (MCE) is satisfied.

The last case to consider is when $C_1 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{w}_1}\}$ and $C_2 = \{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}_2}, \mathbf{x}^{\mathbf{w}_2}\}$ are in $\mathcal{C}_d$, where among all of we exponents we only allow $\mathbf{v}_1$ and $\mathbf{v}_2$ to be equal. Assume first that $\mathbf{v}_1 \neq \mathbf{v}_2$. Without loss of generality let us assume that we want $\mathbf{x}^{\mathbf{v}_1}$ to be in our new circuit and let us consider the set $C_3 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_2\}$. From the definition of $\mathcal{C}_d$ either $C_3 \in \mathcal{C}_d$ or a two element subset of C_3 is in $\mathcal{C}_d$. Notice that $\{\mathbf{x}^{\mathbf{v}_2}, \mathbf{x}^{\mathbf{w}_2}\}$ is not in $\mathcal{C}_d$. In all other cases we have $\mathbf{x}^{\mathbf{v}_1}$ is in an allowed subset of C_3. Assume now $\mathbf{v}_1 = \mathbf{v}_2$. We want $\mathbf{x}^{\mathbf{w}_1}$ to be in our new circuit. Let us consider the set $C_3 = \{\mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{w}_1}, \mathbf{x}^{\mathbf{w}_2}\}$. Since we cannot have $\mathbf{v}_1 - \mathbf{w}_1$ or $\mathbf{v_1} - \mathbf{w}_2$ in A, either $C_3 \in \mathcal{C}_d$ or $\{\mathbf{x}^{\mathbf{w}_1}, \mathbf{x}^{\mathbf{w}_2}\} \in \mathcal{C}_d$. In both cases $\mathbf{x}^{\mathbf{w}_1} \in C_3$ so axiom (MCE) is satisfied.

So $\mathcal{C}_d$ is the set of circuits for a matroid M_d for each d. By construction, each matroid M_d has rank two. Namely, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for the lattice L then binomials $\mathbf{x}^{\mathbf{u}_1} \oplus \mathbf{x}^{\mathbf{v}_1}$ and $\mathbf{x}^{\mathbf{u}_2} \oplus \mathbf{x}^{\mathbf{v}_2}$, where $\mathbf{u}_1 - \mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{u}_2 - \mathbf{v}_2 = \mathbf{e}_2$, cannot be in J_d at the same time. Since there are no monomials in $\mathcal{C}_d$, it means that at least one of $\{\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{v}_1}\}$ and $\{\mathbf{x}^{\mathbf{u}_2}, \mathbf{x}^{\mathbf{v}_2}\}$ is an independent set. There are no bigger independent sets in $\mathcal{C}_d$ as any three monomials are defined to be dependent.

Let us denote by J_d the set of all polynomials in $\mathbb{B}[x_1, \ldots, x_n]$ such that their support is either a circuit of M_d or a union of circuits. We will show that J_d are homogeneous parts of one ideal $J \in S$, i.e., that if $p \in J_d$ then $x_i p \in J_{d+1}$ for $1 \leq i \leq n$. We have that $p = m_1 \oplus \cdots \oplus m_r$, where m_i are monomials of degree d. The polynomial p can be written as $p = p_1 \oplus \cdots \oplus p_s$, for some $s > 0$, where p_j are polynomials in J of minimal support. So it is enough to check this property for polynomials of minimal support in J. Let us first assume that $p = \mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{v}}$. Since $p \in J_d$, we have that $\mathbf{u} - \mathbf{v} \in A$. For $x_i p = \mathbf{x}^{\mathbf{u}'} \oplus \mathbf{x}^{\mathbf{v}'}$ we have $\mathbf{u} - \mathbf{v} = \mathbf{u}' - \mathbf{v}' \in A$ and it follows that $x_i p \in J_{d+1}$. If $p = \mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{v}} \oplus \mathbf{x}^{\mathbf{w}}$ is of minimal support, it means that none of $\mathbf{u} - \mathbf{v}$, $\mathbf{u} - \mathbf{w}$, $\mathbf{v} - \mathbf{w}$ is in A. If $x_i p = \mathbf{x}^{\mathbf{u}'} \oplus \mathbf{x}^{\mathbf{v}'} \oplus \mathbf{x}^{\mathbf{w}'}$ then we have $\mathbf{u} - \mathbf{v} = \mathbf{u}' - \mathbf{v}'$, $\mathbf{u} - \mathbf{w} = \mathbf{u}' - \mathbf{w}'$, $\mathbf{v} - \mathbf{w} = \mathbf{v}' - \mathbf{w}'$ and in particular none of $\mathbf{u}' - \mathbf{v}'$, $\mathbf{u}' - \mathbf{w}'$, $\mathbf{v}' - \mathbf{w}'$ is in A. So $x_i p$ is a polynomial of minimal support in J_{d+1}.

It remains to show that the constructed ideal is saturated. We define $p = x_i(m_1 \oplus \cdots \oplus m_r)$ to be a polynomial in J_d, where m_i are monomials of degree $d - 1$. The polynomial p can be written as a sum of polynomials p_j of minimal support in J_d, where each p_j is of the form $x_i p_j'$ for some homogeneous polynomial p_j' of degree $d - 1$. This means that if we show that for each $x_i p \in J_d$ of minimal support we have $p \in J_{d-1}$ then we are done. So let us assume that $p = x_i(\mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{v}} \oplus \mathbf{x}^{\mathbf{w}}) = \mathbf{x}^{\mathbf{u}'} \oplus \mathbf{x}^{\mathbf{v}'} \oplus \mathbf{x}^{\mathbf{w}'}$ is of minimal support in J_d. This means that none of $\mathbf{u}' - \mathbf{v}'$, $\mathbf{u}' - \mathbf{w}'$, $\mathbf{v}' - \mathbf{w}'$ is in A. But since $\mathbf{u} - \mathbf{v} = \mathbf{u}' - \mathbf{v}'$, $\mathbf{u} - \mathbf{w} = \mathbf{u}' - \mathbf{w}'$, $\mathbf{v} - \mathbf{w} = \mathbf{v}' - \mathbf{w}'$ it follows that none of $\mathbf{u} - \mathbf{v}$, $\mathbf{u} - \mathbf{w}$, $\mathbf{v} - \mathbf{w}$ is in A and so $\mathbf{x}^{\mathbf{u}} \oplus \mathbf{x}^{\mathbf{v}} \oplus \mathbf{x}^{\mathbf{w}}$ is of minimal support in J_{d-1}. By the same argument we prove that if $p = x_i(\mathbf{x^u} \oplus \mathbf{x^v}) \in J_d$ then $\mathbf{x^u} \oplus \mathbf{x^v} \in J_{d-1}$. $\square$

Corollary 4.2.3. *Let us define $\mathcal{C}_d$ to be the set whose elements are all triples of monomials of degree d in $\mathbb{B}[x_1, \ldots, x_n]$. Then the set $\cup_1^\infty \mathcal{C}_d$ is the set of supports of all homogeneous polynomials of minimal support of an ideal $J \in \mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$.*

Proof. Follows from the proof of Lemma 4.2.2 if we set $A = \emptyset$. $\square$

We are now ready to state and prove the main theorem of this chapter. In particular, we will see that if J is an ideal in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$ and A is a proper sublattice of L then the two constructions introduced above are inverses of each other.

Theorem 4.2.4. *Let L be a lattice of all integer points in $\mathbb{Z}^n$ in the hyperplane defined by $x_1 + x_2 + \cdots + x_n = 0$. There is a one to one correspondence between tropical ideals in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$ and proper sublattices of L.*

Proof. We have seen in Lemma 4.2.2 that every proper sublattice of L gives a tropical ideal in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$. By construction, all these ideals are different.

It remains to show that if J is a tropical ideal in $\mathcal{I}_{\mathrm{n}}^{\mathrm{tr},2,\mathbb{B}}$ then there exists a sublattice $A \subset L$ such that for $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$, $\mathbf{u} - \mathbf{v} \in A$ if and only if $\mathbf{x^u} \oplus \mathbf{x^v}$ is a binomial in J.

The case when J does not have any binomials was dealt with in Corollary 4.2.3. So let us assume that J has at least one binomial. Let us consider a sublattice A of L generated by the lattice vectors coming from all homogeneous binomials in J. We have to prove that for every $\mathbf{a} \in A$ there exists a homogeneous binomial $\mathbf{x^u} \oplus \mathbf{x^v}$ in J, such that $\mathbf{u} - \mathbf{v} = \mathbf{a}$. We will prove this in two steps, using induction in both cases.

First let us show that if for some $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$

$$p = \mathbf{x^u} \oplus \mathbf{x^v}$$

is a homogeneous binomial in J then for any $k \in \mathbb{Z}^*$ there is a polynomial $p' = \mathbf{x^{u'}} \oplus \mathbf{x^{v'}}$ in J such that $\mathbf{u'} - \mathbf{v'} = k(\mathbf{u} - \mathbf{v})$. For an inductive step let us assume that the binomial

$$q = \mathbf{x^s} \oplus \mathbf{x^t},$$

where $\mathbf{s} - \mathbf{t} = (k-1)(\mathbf{u} - \mathbf{v})$, is in J. Then polynomials

$$\mathbf{x^u} q = \mathbf{x^{u+s}} \oplus \mathbf{x^{u+t}} \quad \text{and} \quad \mathbf{x^t} p = \mathbf{x^{t+u}} \oplus \mathbf{x^{t+v}}$$

are also in J, so it follows that

$$\mathbf{x}^{\mathbf{u}+\mathbf{s}} \oplus \mathbf{x}^{\mathbf{t}+\mathbf{v}}$$

is in J, where

$$(\mathbf{u}+\mathbf{s}) - (\mathbf{t}+\mathbf{v}) = (\mathbf{u}-\mathbf{v}) + (\mathbf{s}-\mathbf{t}) = k(\mathbf{u}-\mathbf{v}).$$

The second induction is on the number of binomials. Let $Q = \{\mathbf{x}^{\mathbf{u}} + \mathbf{x}^{\mathbf{v}} \mid |\mathbf{u}| = |\mathbf{v}|\}$ be a finite set of homogeneous binomials in J. Let $\mathbf{a} \in L$ be a non-zero integer combination of vectors $\{\mathbf{u}-\mathbf{v} \mid \mathbf{x}^{\mathbf{u}} + \mathbf{x}^{\mathbf{v}} \in Q\}$ in L. For an inductive step let us assume that there are vectors $\mathbf{u}_a, \mathbf{v}_a \in \mathbb{Z}^n$ and a binomial

$$p_a = \mathbf{x}^{\mathbf{u}_a} \oplus \mathbf{x}^{\mathbf{v}_a}$$

in J such that $\mathbf{u}_a - \mathbf{v}_a = \mathbf{a}$. Let $p_j = \mathbf{x}^{\mathbf{u}_j} \oplus \mathbf{x}^{\mathbf{v}_j}$ be a binomial in J which is not in Q. Let us fix $l \in \mathbb{Z}^*$ and consider $\mathbf{a}_0 = \mathbf{a} + l(\mathbf{u}_j - \mathbf{v}_j) \in L$. By the first part of the proof we know that there is a binomial

$$p_q = \mathbf{x}^{\mathbf{u}_q} \oplus \mathbf{x}^{\mathbf{v}_q}$$

in J, such that $\mathbf{u}_q - \mathbf{v}_q = l(\mathbf{u}_j - \mathbf{v}_j)$. We will show now that there exists a binomial $p_0 = \mathbf{x}^{\mathbf{u}_0} \oplus \mathbf{x}^{\mathbf{v}_0} \in J$ such that $\mathbf{u}_0 - \mathbf{v}_0 = \mathbf{a}_0$.

Let us consider two polynomials in J:

$$\mathbf{x}^{\mathbf{u}_q} p_a = \mathbf{x}^{\mathbf{u}_q+\mathbf{u}_a} \oplus \mathbf{x}^{\mathbf{u}_q+\mathbf{v}_a}$$

and

$$\mathbf{x}^{\mathbf{v}_a} p_q = \mathbf{x}^{\mathbf{v}_a+\mathbf{u}_q} \oplus \mathbf{x}^{\mathbf{v}_a+\mathbf{v}_q}.$$

It follows that a polynomial $\mathbf{x}^{\mathbf{u}_q+\mathbf{u}_a} \oplus \mathbf{x}^{\mathbf{v}_a+\mathbf{v}_q}$ is also in J and

$$(\mathbf{u}_q+\mathbf{u}_a) - (\mathbf{v}_a+\mathbf{v}_q) = (\mathbf{u}_a-\mathbf{v}_a) + (\mathbf{u}_q-\mathbf{v}_q) = \mathbf{a} + l(\mathbf{u}_j-\mathbf{v}_j) = \mathbf{a}_0.$$

The statement follows by induction. □

Let us come back to Table 4.1. Using Theorem 4.2.4 we see that all the cases but 2, 3 and 9 give degree two parts of tropical ideals and that for 5, 8, 10, 12, 13, 14 and 15 we have freedom in choosing some of the binomials in higher degrees.

Example 4.2.5. There are infinitely many tropical ideals whose degree two part corresponds to the case 5 from Table 4.1. The vector $(2, -1, -1)$ is one of the generators of the sublattice. We can add to the list of generators any vector of the form $(k, l, -k - l)$, where $k, l \in \mathbb{Z}$, as long as $(k, l, -k - l)$ and $(2, -1, -1)$ do not span the lattice L. There is also one tropical ideal whose all binomials come from the integers multiples of $(2, -1, -1)$.

Chapter 5

Realizability of tropical ideals with Hilbert function two

As we discussed before, a tropical ideal $J \subset \bar{\mathbb{R}}[x_1, \ldots, x_n]$ is realizable if there exists a field K and an ideal $I \subset K[x_1, \ldots, x_n]$ such that $\operatorname{trop}(I) = J$. In this chapter we will discuss realizability of saturated tropical ideals with Hilbert function two in two and three variables and with coefficients in the field $\mathbb{B}$, i.e., ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ and $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$.

5.1 Ideals in two variables

In this section we will show that all ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ are realizable.

Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}} \subset \mathbb{B}[x, y]$. Let us assume that we know the degree d of J, which we denote by J_d. Then by multiplying all the polynomials in J_d by x and by y, we know the dependency between all pairs of monomials in degree $d + 1$ of J except $\{x^{d+1}, y^{d+1}\}$. So to describe an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ it is enough if for each degree d we say whether the set $\{x^d, y^d\}$ is dependent or not. In fact, as the next proposition shows, we can do much better.

Proposition 5.1.1. *Let $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}} \subset \mathbb{B}[x, y]$. Then J is determined by the smallest degree d such that $x^d \oplus y^d \in J$.*

Proof. From Theorem 4.2.4 we know that for the lattice $L = \{l(1, -1) \mid l \in \mathbb{Z}\}$, each sublattice of L which is spanned by $(k, -k)$, where $k > 1$, corresponds to a different ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$. If a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ corresponds to a lattice spanned by $(d, -d)$ then $x^d \oplus y^d$ is the binomial of the lowest degree in J. Also, there is exactly one ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ which does not have any binomials. $\square$

Proposition 5.1.1 tells us that every tropical ideal $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\mathbb{B}}$ that contains a binomial is determined by the smallest d such that $x^d \oplus y^d \in J$. For a given d let us denote the corresponding ideal by J^d. If no such d exists, i.e., J has no binomials, then let us write $J = J^\infty$. The next two lemmas show that every tropical ideal of the form J^d or J^∞ is realizable over $\mathbb{C}[x, y]$, where $\mathbb{C}$ is equipped with the trivial valuation.

Lemma 5.1.2. *For each $d > 1$ there exists an ideal $I^d \subset \mathbb{C}[x, y]$ such that* $\mathrm{trop}(I^d) = J^d$.

Proof. Let ϵ_{2d} be a primitive $2d$-th root of unity and let $e = (-\epsilon_{2d} - \epsilon_{2d}^{2d-1})$. We will show that the ideal $I^d = \langle x^2 + exy + y^2 \rangle$ is such that $\mathrm{trop}(I^d) = J^d$. Note that the ideal I^d has Hilbert function two and is saturated with respect to xy. So $\mathrm{trop}(I^d) \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\mathbb{B}}$. It is enough to show that the binomial of the lowest degree in I^d has support $\{x^d, y^d\}$.

Let us consider $x^{2d} - y^{2d}$. We have

$$x^{2d} - y^{2d} = \prod_{0 \le j < 2d} \left(x - \epsilon_{2d}^j y \right),$$

where ϵ_{2d}^j are the $2d$-th roots of unity. So $x^2 + exy + y^2 = (x - \epsilon_{2d} y)(x - \epsilon_{2d}^{2d-1} y)$ divides $x^{2d} - y^{2d}$, i.e., $x^{2d} - y^{2d} \in I^d$. But we also have $x^{2d} - y^{2d} = (x^d + y^d)(x^d - y^d)$ and the factor $x^d + y^d$ contains all factors $(x - \epsilon_{2d}^i y)$ with i odd. In particular, all factors corresponding to the primitive $2d$-th roots of unity are in the factor $x^d + y^d$. As ϵ_{2d} and ϵ_{2d}^{2d-1} are primitive, it means $x^2 + exy + y^2$ divides $x^d + y^d$, i.e., $x^d + y^d \in I^d$. So we have $x^d \oplus y^d \in \mathrm{trop}(I^d)$.

Let us assume that there exists some $1 < m < d$ and $\alpha \in \mathbb{C}$ such that $x^m - \alpha y^m \in I^d$. We have

$$x^m - \alpha y^m = \prod_{0 \le j < m} \left(x - \alpha_m \epsilon_m^j y \right),$$

where ϵ_m^j are the m-th roots of unity and $\alpha_m = \sqrt[m]{|\alpha|} \exp(\frac{i \arg(\alpha)}{m})$. If $x^m - \alpha y^m \in I^d = \langle (x - \epsilon_{2d} y)(x - \epsilon_{2d}^{2d-1} y) \rangle$ then for some $k, l \in \mathbb{N} \cup \{0\}$, where $k, l < m$, we must have

$$\begin{aligned} \epsilon_{2d} &= \alpha_m \epsilon_m^k, \\ \epsilon_{2d}^{-1} &= \alpha_m \epsilon_m^l. \end{aligned}$$

Raising both sides of the above equations to the m-th power we get

$$\epsilon_{2d}^{m} = \alpha,$$
$$\epsilon_{2d}^{-m} = \alpha,$$

which gives

$$\epsilon_{2d}^{2m} = 1.$$

As $1 < m < d$ and ϵ_{2d} is a $2d$-th primitive root of unity, this is not possible. So $x^d \oplus y^d \notin \operatorname{trop}(I^d)$ for any $m < d$. $\square$

Note that under the change of variables $x = v$, $y = \mu w$, where $\mu \neq 0$, the ideal I^d from the above proof becomes $I^d_{vw} = \langle v^2 + e\mu vw + \mu^2 w^2 \rangle \subset \mathbb{C}[v, w]$. This gives us the following result.

Corollary 5.1.3. *Let ϵ_{2d} be a primitive $2d$-th root of unity and let $e = (-\epsilon_{2d} - \epsilon_{2d}^{2d-1})$. Then for any $\mu \in \mathbb{C}^*$ the ideal $I^d_{vw} = \langle v^2 + e\mu vw + \mu^2 w^2 \rangle$ satisfies $\operatorname{trop}(I^d) = J^d$.*

Proof. We have to show that the ideal I^d_{vw} has the property that a binomial of smallest degree in I^d_{vw} has support $\{v^d, w^d\}$. Let us note that under the change of coordinates $x = v$, $y = \mu w$ we have $v^d + \mu^d w^d = x^d + y^d$. So I^d_{vw} has a binomial of degree d. If I^d_{vw} had a binomial for some $d' < d$, this would imply that $v^{d'} + \eta w^{d'} = x^{d'} + \frac{\eta}{\mu^{d'}} x^{d'} \in I^d$ for some $\eta \in \mathbb{C}^*$, which is a contradiction. $\square$

Lemma 5.1.4. *There exists an ideal $I \subset \mathbb{C}[x, y]$ such that $\operatorname{trop}(I) = J^\infty$.*

Proof. We want to find an ideal $I \subset \mathbb{C}[x, y]$ with constant Hilbert function two and saturated with respect to xy such that for any d the underlying matroid of I_d is the uniform matroid $\mathcal{U}^2_{d+1}$. It means that we need to find a pair $(e, f) \in (\mathbb{C}^*)^2$ such that all homogeneous polynomials of minimal support in $I = \langle x^2 + exy + fy^2 \rangle$ have support of size three. For each degree d this condition gives us a finite number of polynomial equations in e and f. The union of the zero sets of these polynomials has measure zero in $(\mathbb{C}^*)^2$. If we take the union of these sets for all degrees then we have the union of countably many sets of measure zero which is again of measure zero. So the complement of this set is non empty and so such a pair (e, f) exists and we are done. $\square$

Theorem 5.1.5. *Every tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ is realizable.*

Proof. By Proposition 5.1.1, every tropical ideal $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ is of the form J^d for some $d \in \mathbb{N} \setminus \{1\} \cup \{\infty\}$. By Lemma 5.1.2, J^d is realizable for any $d \in \mathbb{N} \setminus \{1\}$. The ideal J^∞ is realizable by Lemma 5.1.4. $\square$

5.2 Ideals in three variables

Here we will study realizability of saturated tropical ideals in three variables with Hilbert function two, i.e., ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}} \subset \mathbb{B}[x,y,z]$. First we will focus on ideals whose degree one part has a binomial and show that all of these ideals are realizable over $\mathbb{C}[x,y,z]$, where $\mathbb{C}$ is equipped with the trivial valuation. After that we shift our attention to the ideals whose degree one part is $\{x \oplus y \oplus z\}$. Here we give partial results showing that many of these ideals are realizable. However there are still some unresolved cases.

5.2.1 Ideals with a circuit of size two in degree one

Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}} \subset \mathbb{B}[x,y,z]$ with a binomial in its degree one part. Without loss of generality we can assume that the binomial is of the form $x \oplus z$. Before showing that all of the above ideals are realizable we need to introduce two lemmas.

Lemma 5.2.1. *Let $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ be such that $J_1 = \{x \oplus z\}$ and let $J' := J \cap \mathbb{B}[x,y]$. Then J' is a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$.*

Proof. By Lemma 2.4.11, J' is a tropical ideal in $\mathbb{B}[x,y]$. Since J is saturated with respect to xyz it follows that J' is saturated with respect to xy. We have to show that Hilbert function of J' is two. Note that for each degree d, the corresponding matroid of J'_d does not have independent sets of size greater than two or dependent sets of size one. If there were no independent sets of size two it would mean that any two monomials form a circuit. In particular, $\{x^d, x^{d-1}y\}$ would be a circuit in J'_d, and so also in J_d. By saturation, we would have that $x \oplus y \in J_1$. Since we already have $x \oplus z \in J_1$, this would mean that the underlying matroid of J_1 has rank one, which is a contradiction. □

Lemma 5.2.2. *Let $\tilde{J} \subset \mathbb{B}[x,y]$ be an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$. There exists exactly one tropical ideal $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}} \subset \mathbb{B}[x,y,z]$ such that $x \oplus z \in J$ and $J \cap \mathbb{B}[x,y] = \tilde{J}$.*

Proof. We know from Proposition 5.1.1 that the ideal $\tilde{J}$ is determined by the smallest $d \in \{\mathbb{N} \cup \infty\}$ such that $x^d \oplus y^d$ is in $\tilde{J}$ so, as in the previous section, we can denote it by $\tilde{J}^d$.

We claim that the ideal J is given by a sublattice of $L = \{l_1(1,0,-1) + l_2(0,1,-1) \mid l_1, l_2 \in \mathbb{Z}\}$ generated by $(1,0,-1)$, $(d,-d,0)$ if $\tilde{J} = \tilde{J}^d$ for $d \in \mathbb{N}$ and by $(1,0,-1)$ if $\tilde{J} = \tilde{J}^\infty$. In order to prove this, we need to show that in both cases $J \cap \mathbb{B}[x,y] = \tilde{J}$ and that there exists no other ideal $J' \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ such that $x \oplus z \in J'$ and $J' \cap \mathbb{B}[x,y] = \tilde{J}$.

By construction, we have that $x \oplus z \in J$. Let us first consider the case $\tilde{J} = \tilde{J}^d$, where $d \in \mathbb{N}$. We have $(d, -d, 0) \in J$. Also, note that for any $d' < d$, $(d', -d', 0)$ is not the integer combination of $(1, 0, -1)$ and $(d, -d, 0)$. So $x^{d'} \oplus y^{d'}$ is not in J for any $d' < d$ and $J \cap \mathbb{B}[x, y] = \tilde{J}$ indeed. In the case when $\tilde{J} = \tilde{J}^\infty$ we have that for every d', $(d', -d', 0)$ is not in the lattice spanned by $(1, 0, -1)$ so we again get that $J \cap \mathbb{B}[x, y] = \tilde{J}$.

What is left to show is that there is no other ideal J' satisfying the above conditions. First consider the case when $\tilde{J} = \tilde{J}^\infty$. We know that $(1, 0, -1)$ must be one of the generators of the lattice for J'. Assume that $(a, b, -(a + b))$ is another generator. Since $J' \cap \mathbb{B}[x, y] = \tilde{J}^\infty$ we must have that $a + b \neq 0$ and so consider the integer combination of the generators

$$\begin{aligned} -(a+b)(1,0,-1) + (a,b,-a-b) &= (-a-b,0,a+b) + (a,b,-a-b) \\ &= (-b,b,0). \end{aligned}$$

Since we have $J' \cap \mathbb{B}[x, y] = \tilde{J}^\infty$ this implies we must have $b = 0$. This means we have

$$(a, b, -a - b) = (a, 0, -a) = a(1, 0, -1),$$

so $(a, b, -a - b)$ was not another generator for the sublattice. Now assume $\tilde{J} = \tilde{J}^d$, where $d \neq \infty$. The vector $(1, 0, -1)$ must be one of the generators and let us assume that $(a, b, -(a + b))$ is another generator of J'. Since $x^d \oplus y^d \in \tilde{J}^d$, we also have $x^d \oplus y^d \in J'$ and so $(d, -d, 0)$ must be in the lattice generated by $(1, 0, -1)$ and $(a, b, -(a + b))$. So there exists some $k, l \in \mathbb{Z}$ such that

$$k(1, 0, -1) + l(a, b, -(a + b)) = (d, -d, 0).$$

This means we must have $d = -lb$, so d must be divisible by b. Assume that $a + b = 0$. Then

$$(a, b, -(a + b)) = (-b, b, 0)$$

and so a polynomial $x^b \oplus y^b$ is in $\tilde{J}^d$. Since $b \mid d$ and d was the smallest degree such that $x^d \oplus y^d$ is in J' this is possible only if $b = d$. Assume now that $a + b \neq 0$ and consider

$$(a + b)(1, 0, -1) - (a, b, -(a + b)) = (b, -b, 0).$$

We again have that $x^b \oplus y^b$ is in $\tilde{J}^d$ only in $b = d$, which finishes the proof.

□

Theorem 5.2.3. *All tropical ideals in $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ that have a binomial in their* *degree one part are realizable over* $\mathbb{C}[x, y, z]$.

Proof. Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ with a circuit of size two in its degree one part. Without loss of generality we can assume that $x \oplus z \in J_1$. Let us consider $J' = J \cap \mathbb{B}[x, y]$. By Lemma 5.2.1, J' is a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ and we saw in Section 5.1 that all such ideals are realizable over $\mathbb{C}[x, y, z]$. Let $I' := \langle x^2 + \alpha xy + \beta y^2\rangle \subset \mathbb{C}[x, y]$ be a realization of J' for some $\alpha, \beta \in \mathbb{C}$. Let us consider the ideal $I = \langle x - z, x^2 + \alpha xy + \beta y^2\rangle \subset \mathbb{C}[x, y, z]$. We will show that $\mathrm{trop}(I) = J$. Note that $I \cap \mathbb{C}[x, y] = I'$, since the generators of I form a Gröbner basis under the lexicographic order with $z > y > x$. Using Lemma 2.4.13 we get

$$J \cap \mathbb{B}[x, y] = J' = \mathrm{trop}(I') = \mathrm{trop}(I \cap \mathbb{C}[x, y]) = \mathrm{trop}(I) \cap \mathbb{B}[x, y].$$

Since both J and $\mathrm{trop}(I)$ are tropical ideals in $\mathbb{B}[x, y, z]$ with Hilbert function two and degree one part $x \oplus z$, the fact that $J \cap \mathbb{B}[x, y] = \mathrm{trop}(I) \cap \mathbb{B}[x, y]$ together with Lemma 5.2.2 finishes the proof.

□

5.2.2 Ideals with a circuit of size three in degree one

Before studying realizability of tropical ideals without binomials in their degree one part, let us focus on the classical side for the moment. Let I be an ideal in $\mathbb{C}[x, y, z]$ generated by $\langle x + by + cz, x^2 + exy + fy^2\rangle$, where $bcf \neq 0$. The generators form a Gröbner basis under the lexicographic order with $z > y > x$ so we have

$$I \cap \mathbb{C}[x, y] = \langle x^2 + exy + fy^2\rangle.$$

Since $y = \frac{-x-cz}{b}$ and $b \neq 0$, the ideal I can be written equivalently in the form

$$I = \langle x + by + cz, \frac{b^2 - be + f}{c^2 f} x^2 + \frac{2f - be}{cf} xz + z^2\rangle.$$

Using the lexicographic order with $y > z > x$ and $x > y > z$, correspondingly, we get

$$I \cap \mathbb{C}[x, z] = \langle \frac{b^2 - be + f}{c^2 f} x^2 + \frac{2f - be}{cf} xz + z^2\rangle.$$

Note that both ideals $I \cap \mathbb{C}[x, y]$ and $I \cap \mathbb{C}[x, z]$ are saturated and with Hilbert function two. Let $d_{xz} := \frac{b^2-be+f}{c^2 f}$ and $e_{xz} := \frac{2f-be}{cf}$.

Lemma 5.2.4. *Let* $J_{xy} \subset \mathbb{B}[x, y]$ *and* $J_{xz} \subset \mathbb{B}[x, z]$ *be ideals in* $\mathcal{I}^{\mathrm{tr},\mathrm{k},\mathbb{B}}_2$ *such that at least one of* J_{xy} *and* J_{xz} *contains a binomial. Then there exists exactly one tropical* *ideal* $J \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}} \subset \mathbb{B}[x,y,z]$ *such that* $J \cap \mathbb{B}[x,y] = J_{xy}$ *and* $J \cap \mathbb{B}[x,z] = J_{xz}$.

Proof. We know from Proposition 5.1.1 that the ideals J_{xy} and J_{xz} are determined by the smallest $d_1, d_2 \in \{\mathbb{N} \cup \infty\}$ such that $x^{d_1} \oplus y^{d_1}$ is in J_{xy} and $x^{d_2} \oplus z^{d_2}$ is in J_{xz}. As in the previous section, we can denote them by $J^{d_1}_{xy}$ and $J^{d_2}_{xz}$.

Without loss of generality let us assume that $J^{d_2}_{xz}$ has a binomial, i.e. that $d_2 \neq \infty$. We claim that the ideal J is given by a sublattice A of the lattice $L = \{l_1(1,0,-1) + l_2(0,1,-1) \mid l_1, l_2 \in \mathbb{Z}\}$ which is generated by $(d_2, 0, -d_2)$, $(d_1, -d_1, 0)$ if $J_{xy} = J^{d_1}_{xy}$ for $d_1 \neq \infty$ or by $(d_2, 0, -d_2)$ if $J_{xy} = J^{\infty}_{xy}$. In order to prove this, we need to show that in both cases $J \cap \mathbb{B}[x,y] = J^{d_1}_{xy}$ and $J \cap \mathbb{B}[x,z] = J^{d_2}_{xz}$, and that there exists no other ideal $J' \in \mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ with these properties.

By construction, we have that $x^{d_2} \oplus z^{d_2} \in J$. Let us first consider the case $J_{xy} = J^{d_1}_{xy}$, where $d_1 \in \mathbb{N}$. Then $(d_1, -d_1, 0) \in J$. Also, note that for any $d'_1 < d_1$, $(d'_1, -d'_1, 0)$ is not the integer combination of $(d_2, 0, -d_2)$ and $(d_1, -d_1, 0)$. So $x^{d'_1} \oplus y^{d'_1}$ is not in J for any $d'_1 < d_1$ and $J \cap \mathbb{B}[x,y] = J^{d_1}_{xy}$ indeed. In the case when $J_{xy} = J^{\infty}_{xy}$ we have that for every d'_1, $(d'_1, -d'_1, 0)$ is not in the lattice spanned by $(d_2, 0, -d_2)$ so we again get that $J \cap \mathbb{B}[x,y] = J^{\infty}_{xy}$.

What is left to show is that there is no other ideal J' satisfying the above conditions. First consider the case when $J_{xy} = J^{\infty}_{xy}$. We will start from showing that $(d_2, 0, -d_2)$ must be one of the generators of the sublattice A' for J'. Note that $J \cap \mathbb{B}[x,y] = J^{\infty}_{xy}$ and $J \cap \mathbb{B}[x,z] = J^{d_2}_{xz}$ implies that $J \cap \mathbb{B}[y,z] = J^{\infty}_{yz}$. For a contradiction, assume that $J \cap \mathbb{B}[y,z] = J^{d'}_{yz}$, for some $d' \in \mathbb{N}$. Then both $(d_2, 0, -d_2)$ and $(0, d', -d')$ are in the sublattice A' and so also

$$d'(d_2, 0, -d_2) - d_2(0, d', -d') = (d'd_2, -d'd_2, 0)$$

is in the sublattice. This is a contradiction, since $J \cap \mathbb{B}[x,y] = J^{\infty}_{xy}$. Assume now $(a, b, -a-b)$ and $(u, v, -u-v)$, where $a, b, u, v \in \mathbb{Z}$, are generators for the sublattice A'. Since $J \cap \mathbb{B}[x,y] = J^{\infty}_{xy}$ we must have that $a + b \neq 0$ and $u + v \neq 0$. But then

$$(u+v)(a, b, -a-b) - (a+b)(u, v, -u-v) = (av - bu, -(av - bu), 0)$$

is in the sublattice A' which is possible only if $av = bu$, that is, when $(a, b, -a-b)$ and $(u, v, -u-v)$ are dependent. So the sublattice A' must have only one generator, say $(a, b, -a-b)$. Since $(d_2, 0, -d_2)$ is in the sublattice we have $b = 0$ and

$$(d_2, 0, -d_2) = x(a, 0, -a)$$

for some $x \in \mathbb{Z}$. But since d_2 is the smallest degree such that $x^{d_2} \oplus z^{d_2}$ is in $J^{d_2}_{xz}$, we cannot have that $|a| < d_2$. So $(d_2, 0, -d_2)$ must be the generator indeed.

Now assume $J_{xy} = J_{xy}^{d_1}$, where $d_1 \neq \infty$. Let A be the sublattice generated by vectors $\mathbf{a}_1 = (d_1, -d_1, 0)$ and $\mathbf{a}_2 = (d_2, 0, -d_2)$. Let A' be another sublattice of L such that $\mathbf{a}_1, \mathbf{a}_2 \in A'$ and $A \neq A'$. So we have that $A \subset A'$. Let $\mathbf{b}$ be an element of $A' \setminus A$ and let us consider the lattice $\tilde{A}$ generated by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}$. We have that $\tilde{A} \subset A'$. But since $A \subsetneq \tilde{A}$ and A has full rank, we must have $\tilde{A} = A' = L$. So we must have $A = A'$, which finishes the proof. $\square$

Theorem 5.2.5. *Let $J \subset \mathbb{B}[x, y, z]$ be an ideal in $\mathcal{I}_{\mathrm{x,y,z}}^{\mathrm{tr},2,\mathbb{B}}$ which is given by any of the following sublattices of the lattice L:*

1. *$\{m(k, -k, 0) + n(l, 0, -l) \mid m, n \in \mathbb{Z}\}$, where $k \geq 2, l > 2$,*
2. *$\{m(1, 1, -2) + n(1, -2, 1) \mid m, n \in \mathbb{Z}\}$,*
3. *$\{m(1, 1, -2) + n(2, -2, 0) \mid m, n \in \mathbb{Z}\}$.*

Then J is realizable over $\mathbb{C}[x, y, z]$.

Proof. In all three cases we want to find complex numbers b, c, e, f such that $I = \langle x + by + cz, x^2 + exy + fy^2 \rangle \subset \mathbb{C}[x, y, z]$ is saturated, has Hilbert function two and $\mathrm{trop}(I) = J$. Note that in particular if $bcf \neq 0$ then I has the properties we want.

1. Without loss of generality, let us assume that $k \leq l$. We want to find numbers $b, c, f \in \mathbb{C}^*$ and $e \in \mathbb{C}$ such that the ideal $I = \langle x + by + cz, x^2 + exy + fy^2 \rangle \subset \mathbb{C}[x, y, z]$ tropicalizes to J, i.e., $\mathrm{trop}(I) = J$. We have that

$$I \cap \mathbb{C}[x, y] = \langle x^2 + exy + fy^2 \rangle$$

and

$$I \cap \mathbb{C}[x, z] = \langle d_{xz}x^2 + e_{xz}xz + z^2 \rangle,$$

where

$$d_{xz} = \frac{b^2 - be + f}{c^2 f} \qquad \text{and} \qquad e_{xz} := \frac{2f - be}{cf},$$

are saturated ideals in two variables with Hilbert function two. By Lemma 5.2.4 it is enough to find b, c, e and f such that the binomial of the lowest degree in $I \cap \mathbb{C}[x, y]$ has support $\{x^k, y^k\}$ and the binomial of the lowest degree in $I \cap \mathbb{C}[x, z]$ has support $\{x^l, z^l\}$.

For an arbitrary $f \in \mathbb{C}^*$ let $\hat{f} \in \mathbb{C}^*$ be such that $\hat{f}^2 = f$. It follows from Corollary 5.1.3 that we can set

$$e = \hat{f}(-\epsilon_{2k} - \epsilon_{2k}^{2k-1}).$$

With this choice of e, a binomial with support $\{x^k, y^k\}$ is a binomial of the lowest degree in $I \cap \mathbb{C}[x, y]$. Let us consider $I \cap \mathbb{C}[x, z] = \langle d_{xz}x^2 + e_{xz}xz + z^2 \rangle$. We want $\{x^l, z^l\}$ to be support of the binomial of the lowest degree in $I \cap \mathbb{C}[x, z]$. Since $l > 2$, we first require that d_{xz} and e_{xz} are non-zero, i.e.,

$$b^2 - be + f \neq 0, \tag{5.1}$$

$$2f - be \neq 0. \tag{5.2}$$

Let $\hat{d}_{xz}$ be such that $\hat{d}_{xz}^2 = d_{xz}$. Remember that we have $d_{xz} = \frac{b^2 - be + f}{c^2 f}$. Using Corollary 5.1.3 again, we want $e_{xz} = \hat{d}_{xz}(-\epsilon_{2l} - \epsilon_{2l}^{2l-1})$, i.e.

$$\frac{2f - be}{cf} = \hat{d}_{xz}(-\epsilon_{2l} - \epsilon_{2l}^{2l-1}). \tag{5.3}$$

So a choice for the coefficients b, c, e, f is as follows. Take any $f, c \in \mathbb{C}^*$ and let $\hat{f}$ be such that $\hat{f}^2 = f$. Set

$$e = \hat{f}(-\epsilon_{2k} - \epsilon_{2k}^{2k-1}).$$

Let b_0 be a solution to Equation (5.3). Note that $b_0 \neq 0$ since otherwise, substituting $b = 0$ into Equation (5.3), we get

$$\epsilon_{2l} + \epsilon_{2l}^{2l-1} = \pm 2,$$

which is not possible. We will show that b_0 solves Equation (5.1) if and only if it solves Equation (5.2). Since $l > 2$, we have $-\epsilon_{2l} - \epsilon_{2l}^{2l-1} \neq 0$. So the left hand side of Equation (5.3) is zero if and only if $2f - b_0 e = 0$ and the right hand side is zero if and only if $\hat{d}_{xz} = 0$, i.e. $b_0^2 - b_0 e + f = 0$.

But we cannot have that Equations (5.1) and (5.2) hold at the same time. To see this, note that solving

$$b^2 - be + f = 0 \qquad \text{and} \qquad 2f - be = 0$$

for b, we get that we must have $4f = e^2$. But remember that $e = \hat{f}(-\epsilon_{2k} - \epsilon_{2k}^{2k-1})$, where $\hat{f}^2 = f$, so this is not possible.

So we set $b = b_0$ and we are done.

2. We want a binomial of the form $z^2 + rxy$, where $r \neq 0$, to be in I. Using the

degree one of I, we can write $z = -\frac{1}{c}x - \frac{b}{c}y$ and it follows

$$z^2 + rxy = \frac{1}{c^2}x^2 + (\frac{2b}{c^2} + r)xy + \frac{b^2}{c^2}y^2.$$

So

$$c^2(z^2 + rxy) = x^2 + (2b + c^2r)xy + b^2y^2$$

and we require $e = 2b + c^2r$, $f = b^2$ and $b \neq 0$. We also want a binomial of the form $y^2 + sxz$, where $s \neq 0$, to be in I. Using equality $z = -\frac{1}{c}x - \frac{b}{c}y$ again, we have

$$y^2 + sxz = y^2 - \frac{bs}{c}xy - \frac{s}{c}x^2$$

so

$$-\frac{c}{s}(y^2 + sxz) = x^2 + bxy - \frac{c}{s}y^2.$$

For this to hold we require that $e = b$, $f = -\frac{c}{s}$ and $c \neq 0$.

So we need to find a solution of the system of equations

$$\begin{cases} e &= 2b + c^2r \\ f &= b^2 \\ e &= b \\ f &= -\frac{c}{s}, \end{cases} \tag{5.4}$$

for $f \in \mathbb{C}$, $b, c, e \in \mathbb{C}^*$. Let us set $r = r_0$ and $s = s_0$, where $r_0, s_0 \in \mathbb{C}^*$ are arbitrary. Then the following is a solution of the system of equations (5.4):

$$\begin{aligned} r &= r_0, \\ s &= s_0, \\ b &= -\frac{1}{r_0^{1/3}s_0^{2/3}}, \\ c &= -\frac{1}{r_0^{2/3}s_0^{1/3}}, \\ e &= -\frac{1}{r_0^{1/3}s_0^{2/3}}, \\ f &= \frac{1}{r_0^{2/3}s_0^{4/3}}. \end{aligned}$$

The corresponding ideal is a realization of J.

3. To have a binomial with support $\{x^2, y^2\}$ in I we need to have $e = 0$. We also

need $z^2 + rxy$ to be in I, for some $r \in \mathbb{C}^*$. Using part 2, for this we require $e = 2b + c^2 r$ and $f = b^2$. So putting the conditions together we must solve

$$\begin{cases} 2b + c^2 r = 0 \\ f = b^2 \end{cases}$$

for $b, c, f \in \mathbb{C}^*$. So to find an ideal I, we set $e = 0$ and for arbitrary $c, r \in \mathbb{C}^*$

$$\begin{aligned} b &= -\frac{c^2 r}{2}, \\ f &= b^2. \end{aligned}$$

It follows the binomials $x^2 + b^2 y^2$ and $z^2 + rxy$ are in I and so I is a realization of J.

$\square$

The proof of the first case in Theorem 5.2.5 relies on the fact that in the proof of Lemma 5.1.2 we gave an explicit description of classical ideals which are realizations of ideals in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ which contain a binomial. On the contrary, the realizability of the ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ which does not contain any binomial was shown in a non-constructive way. Using these methods we cannot study realizability of an arbitrary ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$.

Example 5.2.6. Let us consider a tropical ideal J in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ such that each homogeneous degree d part of J is a uniform matroid of rank two. In this case even though all of the ideals $J \cap \mathbb{B}[x,y]$, $J \cap \mathbb{B}[x,z]$ and $J \cap \mathbb{B}[y,z]$ do not contain binomials we cannot use Lemma 5.1.4 to show that such an ideal is realizable. This is due to the fact that this way we cannot detect for example the binomials of the form $x^a y^b \oplus z^{a+b}$, where $a, b > 0$.

Example 5.2.7. Let J be a tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ whose all binomials come from a binomial $x^a y^b \oplus z^{a+b}$ for some $a, b > 0$. We have not developed a generic way of dealing with the realizability of such ideals other than solving the corresponding system of equations explicitly, like in the second and third case in Theorem 5.2.5.

5.2.3 A tropical ideal not realizable over a field of characteristic different from two

Let $\tilde{J} \subset \mathbb{B}[x,y,z]$ be an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ given by the sublattice generated by $(2,-2,0)$ and $(2,0,-2)$. We will show that $\tilde{J}$ is realizable only over fields of characteristic two other than $\mathbb{F}_2$.

Lemma 5.2.8. *Let q be a non-negative prime different from 2 and K be a field of characteristic q. The ideal $\tilde{J}$ is not realizable over $K[x, y, z]$.*

Proof. There is only one polynomial in the degree one of $\tilde{J}$: $x \oplus y \oplus z$. The degree two of the ideal $\tilde{J}$ has three polynomials with support of size two: $x^2 \oplus y^2$, $x^2 \oplus z^2$, $y^2 \oplus z^2$. So the classical ideal $I \subset K[x, y, z]$ which tropicalizes to $\tilde{J}$ must have polynomials with supports $\{x, y, z\}$, $\{x^2, y^2\}$, $\{y^2, z^2\}$ and $\{y^2, z^2\}$ among its generators. By performing a linear change of variables, we can assume that

$$p_1 = x + y + z$$

and

$$p_2 = x^2 + ay^2$$

are in I for some $a \in K^*$. By multiplying p_1 by $y + z - x$ we get that

$$(y + z - x)p_1 = y^2 + z^2 + 2yz - x^2 \in I_2$$

and using the fact that $p_2 \in I$ we get

$$(1 + a)y^2 + z^2 + 2yz \in I_2.$$

If $a \neq -1$ then a binomial $z + 2y$ in in I_1, a contradiction. So we must have $a \neq 1$. Remember that we assumed that characteristic of K is not 2. So there is a polynomial in I with support $\{y^2, z^2, yz\}$. As $\{y^2, yz\}$ and $\{yz, z^2\}$ are both independent sets in $\tilde{J}_2$, if $\tilde{J}$ is the tropicalization of I, these sets must also be independent in I. So since we have a polynomial with support $\{y^2, z^2\}$ in I, $(1 + a)y^2 + z^2 + 2yz$ is neither a polynomial of minimal support in I nor a linear combination of such polynomials for any choice of a. □

Proposition 5.2.9. *Let K be a field of characteristic 2 and $K \neq \mathbb{F}_2$. Then $\tilde{J}$ is realizable over $K[x, y, z]$. Moreover, $\tilde{J}$ is not realizable over $\mathbb{F}_2[x, y, z]$.*

Proof. Any classical ideal in $K[x, y, z]$ with constant Hilbert function two, saturated with respect to xyz and with the linear part having support $\{x, y, z\}$ is, up to a change of coordinates, of the form $I = \langle x+y+z, y^2+ayz+bz^2 \rangle$ for some $a \in K$ and $b \in K^*$. Any such ideal contains a polynomial $(x+y+z)^2 = x^2+y^2+z^2$. If we want I to be a realization of $\tilde{J}$ then we must have $a = 0$. Then $y^2 + bz^2$, $x^2 + (b + 1)z^2$ and $bx^2 + (b + 1)y^2$ all belong to I and we also must assume $b + 1 \neq 0$, i.e., $b \neq -1$ (this is not possible if $K = \mathbb{F}_2$). Let us tropicalize I using the trivial valuation. This gives us a tropical ideal, say $\hat{J}$. This ideal is such that $\hat{J}_1 = x \oplus y \oplus z$ and all of

$x^2 \oplus y^2$, $x^2 \oplus z^2$ and $y^2 \oplus z^2$ are in $\hat{J}_2$. In Table 4.1 in Section 4.1 we saw that $\tilde{J}$ is the only tropical ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y,z}}$ with this property so we must have $\hat{J} = \tilde{J}$. $\square$

Chapter 6

Hilbert function two with coefficients

Let mon_d denote all the monomials in $\bar{\mathbb{R}}[x,y]_d$ and let $\mathrm{Mat}(\mathrm{mon}_d, 2)$ be a valuated matroid of rank two with the base set mon_d such that no element of mon_d is a dependent set. In this chapter we will show how to encode such family of matroids into a sequence of numbers in $\bar{\mathbb{R}}$ and we will give necessary conditions for such a sequence to correspond to a tropical ideal in $\bar{\mathbb{R}}[x,y]$ with Hilbert function two and saturated with respect to xy (that is, an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$). We will also state a conjecture about the sufficient conditions for such a sequence to correspond to an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$.

6.1 Necessary conditions

We start this section by studying the valuated circuit elimination axiom (VCE) from Definition 2.2.5 in the case of certain types of circuits.

Lemma 6.1.1. *Let $\mathcal{M}$ be a valuated matroid of rank two on V and let X and Y be valuated circuits in $\mathcal{M}$ such that $2 \leq |\underline{X}| \leq |\underline{Y}|$ and*

$$|\underline{X} \cap \underline{Y}| = \begin{cases} 1, & \text{if } |\underline{X}| = 2 \\ 2, & \text{if } |\underline{X}| = 3. \end{cases}$$

Assume there are $u, v \in V$ such that $X_u = Y_u \neq \infty$ and $X_v < Y_v$. Then there exists a circuit Z in $\mathcal{M}$ such that $Z_u = \infty$, $Z \geq \min(X,Y)$ and $Z_i = \min(X_i, Y_i)$ for all i with $X_i \neq Y_i$.

Proof. Since $\mathcal{M}$ has rank two, the circuits X and Y have support of size two or three. Let us first consider the case when $|\underline{X}| = |\underline{Y}| = 2$. It follows that there are some u, v, w in V such that $X_u = Y_u \neq \infty$, $X_v < Y_v = \infty$ and $Y_w < X_w = \infty$. Since every circuit is also a vector, by (VVE) in Definition 2.2.6, we get that there exists a valuated vector Z in $\mathcal{M}$ such that $Z_u = \infty$, $Z_v = X_v$, $Z_w = Y_w$ and all other entries are ∞. We also note that Z must be a circuit. Otherwise there would be a circuit Z' in $\mathcal{M}$ such that $\underline{Z'} \subsetneq \underline{X}$, which is not possible. So we have that the circuit Z such that $Z_i = \min(X_i, Y_i)$ for all i with $X_i \neq Y_i$ is in $\mathcal{M}$.

Now assume that $|\underline{X}| = 2$ and $|\underline{Y}| = 3$ and that we have $X_u = Y_u \neq \infty$, $X_v < Y_v = \infty$, $Y_w < X_w = \infty$ and $Y_s < X_s = \infty$, for some $u, v, w, s \in V$. Using (VVE) in Definition 2.2.6 we get that there exists a valuated vector Z such that $Z_u = \infty$, $Z_v = X_v$, $Z_w = Y_w$, $Z_s = Y_s$ and all other entries are ∞. Note that Z must be a circuit. Indeed, if Z was a sum of valuated circuits it would force the entries w and s in the $|V|$-tuple to form the support of a circuit in $\mathcal{M}$, a contradiction.

The last case to consider is when $|\underline{X}| = |\underline{Y}| = 3$. So assume we have $X_u = Y_u \neq \infty$, $X_v < Y_v < \infty$, $X_w < Y_w = \infty$, $Y_s < X_s = \infty$. By the valuated circuit elimination axiom (VCE) in Definition 2.2.6 we get that there are circuits Z and Z' in $\mathcal{M}$ such that $Z_v = X_v + v_1$, $Z_w = X_w$, $Z_s = Y_s + s$, $Z'_v = X_v + v_2$, $Z'_w = X_w + w$ and $Z'_s = Y_s$, for some $v_1, v_2, w, s \in \mathbb{R} \cup \{\infty\}$. Note that for Z and Z' to be circuits, we must in fact have that $v_1, v_2, w, s \in \mathbb{R}$. We want to show that $v_1 = w = s = 0$. Assume that $w > 0$ and consider the circuit $\hat{Z} = Z + w\mathbf{1}$. We can use the valuated circuit elimination axiom on $\hat{Z}$ and Z' to get that there is a circuit C in $\mathcal{M}$ such that $C_v \geq \min\{X_v + v_2, X_v + v_1 + w\}$, $C_s = Y_s$ and all other entries are ∞. But this implies that $\underline{C} \subsetneq \underline{Y}$ which cannot happen. So we must have $w = 0$. Similarly we deduce that $s = 0$. Let us now use the valuated circuit elimination axiom on Z and X. We get that a circuit X' is in $\mathcal{M}$ such that $X'_u \geq X_u = Y_u$, $X'_v \geq X_v$, $X'_s = Y_s$ and all other entries are ∞. Comparing X' with Y we deduce that $v_1 = 0$. $\square$

From now on, we will say that we apply the lemma to two polynomials as a short way of saying that we apply it to the underlying valuated circuits.

We have seen that the supports of valuated circuits form the family of circuits of a standard matroid. So it follows that if $J \in \mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$ then J_1 does not have any circuits. The degree two part of J is a valuated matroid of rank two on the set $\{x^2, xy, y^2\}$. Since $\{x^2, xy, y^2\}$ and $\{x^2, y^2\}$ are the only possible circuits in the degree two part of an ideal in $\mathcal{I}^{\mathrm{tr},2,\mathbb{B}}_{\mathrm{x,y}}$ we can assume that the degree two part of J is spanned by

$$x^2 \oplus c_2 \odot xy \oplus e_2 \odot y^2,$$

where $c_2 \in \bar{\mathbb{R}}$, $e_2 \in \mathbb{R}$. If $c_2 \neq \infty$, it follows that the degree three part of J contains the polynomials

$$x^3 \oplus c_2 \odot x^2 y \oplus e_2 \odot xy^2 \quad \text{and}$$
$$c_2 \odot x^2 y \oplus c_2^2 \odot xy^2 \oplus e_2 \odot c_2 \odot y^3$$

and so, by Lemma 6.1.1, also the polynomial

$$x^3 \oplus c_3 \odot xy^2 \oplus e_2 \odot c_2 \odot y^3,$$

where $c_3 = \min\{2c_2, e_2\}$ if $2c_2 \neq e_2$ or $c_3 = s$, for some $s \geq 2c_2$, otherwise. This is why later in this section we will consider the cases $2c_2 = e_2$, $2c_2 > e_2$ and $2c_2 < e_2$ separately.

Whenever it does not cause confusion, we write expressions like $a \odot b \odot x$ as $(a+b)x$. Also by $s = \widehat{\min}\{a, b\}$ we mean that $s = \min\{a, b\}$ if $a \neq b$, and s has some value greater or equal to a if $a = b$.

We will start by studying the properties of a degree d part of an ideal $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$.

Lemma 6.1.2. *Let J be an ideal in $\mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$. Then for each degree d, J has exactly one polynomial $\bar{p}$ such that* $\mathrm{supp}(\bar{p}) \subseteq \{x^d, xy^{d-1}, y^d\}$. *Moreover, $x^d \in$* $\mathrm{supp}(\bar{p})$.

Proof. Since the ideal J has Hilbert function two the set $\{x^d, xy^{d-1}, y^d\}$ must be dependent. The ideal J is saturated so it has no monomials. Since J has no circuits in degree one, we also have that J cannot have a polynomial with support $\{xy^{d-1}, y^d\}$. So possible polynomials in J whose support is a subset of $\{x^d, xy^{d-1}, y^d\}$ have support $\{x^d, xy^{d-1}\}$, $\{x^d, y^d\}$ or $\{x^d, xy^{d-1}, y^d\}$. Note that all these sets contain x^d. We will show that we cannot have more than one polynomial in J with support in $\{x^d, xy^{d-1}, y^d\}$.

Let q_1 and q_2 be polynomials in $\bar{\mathbb{R}}[x, y]$ such that $\mathrm{supp}(q_1) = \{x^d, xy^{d-1}\}$ and $\mathrm{supp}(q_2) = \{x^d, y^d\}$. If q_1 and q_2 were in J then eliminating x^d and using saturation, we would get that a polynomial with support $\{x, y\}$ is in J, which is not true.

So let assume now that polynomials q_1 and p are in J such that $\mathrm{supp}(q_1) = \{x^d, xy^{d-1}\}$ and $\mathrm{supp}(p) = \{x^d, xy^{d-1}, y^d\}$. We have that p is not of minimal support and it means that it is a tropical sum of polynomials of minimal support in J. This implies that there is some $q_2 \in J$ such that $\mathrm{supp}(q_2) = \{x^d, y^d\}$. We saw above that it is impossible. In the same way we show that polynomials p and q_2 cannot both be in J. $\square$

Let $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ and J_d denote the degree d part of J. Let p be a unique polynomial in J_d such that $\operatorname{supp}(p) \subset \{x^d, xy^{d-1}, y^d\}$. Assume that the coefficient in front of x^d in p is 0. From now on we will denote by c_d the coefficient in front of xy^{d-1} and by e_d the coefficient in front of y^d in p. Note that we may have $c_d = \infty$ or $e_d = \infty$. The next two lemmas will describe the coefficients in p in terms of the ones in the polynomials of lower degrees in J.

Lemma 6.1.3. *Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$ and c_2, e_2 unique elements of $\bar{\mathbb{R}}$ such that J_2 is generated by $\{x^2 \oplus c_2xy \oplus e_2y^2\}$. Let $p = x^d \oplus c_dxy^{d-1} \oplus e_dy^d$, where $c_d, e_d \in \bar{\mathbb{R}}$, $d \geq 3$, be a polynomial in J. Then $e_d = e_2 + c_{d-1}$.*

Proof. Assume $c_{d-1} \neq \infty$. Then we know

$$\begin{aligned} x^d \oplus & c_{d-1}x^2y^{d-2} \oplus e_{d-1}xy^{d-1} \quad \text{and} \\ & c_{d-1}x^2y^{d-2} \oplus (c_2 + c_{d-1})xy^{d-1} \oplus (e_2 + c_{d-1})y^d \end{aligned}$$

are in J for some $e_{d-1} \in \bar{\mathbb{R}}$ and are polynomials of minimal support. We use Lemma 6.1.1 on these two polynomials to get that $x^d \oplus c_dxy^{d-1} \oplus (e_2 + c_{d-1})y^d$ is in J, for some $c_d \in \bar{\mathbb{R}}$. Using Lemma 6.1.2, it follows that $e_d = e_2 + c_{d-1}$.

Assume now $c_{d-1} = \infty$. Then we have $x^d \oplus e_{d-1}xy^{d-1} \in J$ for some $e_{d-1} \in \mathbb{R}$, so $e_d = \infty = e_2 + c_{d-1}$. $\square$

Lemma 6.1.4. *Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$ and let c_2, e_2 be unique elements of $\bar{\mathbb{R}}$ such that J_2 is generated by $\{x^2 \oplus c_2xy \oplus e_2y^2\}$. Let $p = a_dx^d \oplus b_dx^{d-1}y \oplus y^d$, where $a_d, b_d \in \bar{\mathbb{R}}$ and $d \geq 3$, be a polynomial in J. Then $a_d = c_{d-1} - (d-1)e_2$ and $b_d = c_d - (d-1)e_2$.*

Proof. First we will show that $a_3 = c_2 - 2e_2$ and $b_3 = c_3 - 2e_2$. If $c_2 \neq \infty$ then the polynomials

$$\begin{aligned} c_2x^3 \oplus 2c_2x^2y \oplus (c_2 + e_2)xy^2 & \quad \text{and} \\ e_2x^2y \oplus (c_2 + e_2)xy^2 \oplus 2e_2y^3 & \end{aligned} \tag{6.1}$$

are polynomials of minimal support in J. By Lemma 6.1.1 also

$$c_2x^3 \oplus b_3x^2y \oplus 2e_2y^3$$

is in J for some $b_3 = \widehat{\min}\{2c_2, e_2\}$. From this we get $a_3 = c_2 - 2e_2$. We can tropically multiply the polynomials in (6.1) by $(-c_2)$ and $(c_2 - e_2)$ and then apply Lemma 6.1.1

to the polynomials

$$x^3 \oplus c_2x^2y \oplus e_2xy^2 \quad \text{and}$$
$$c_2x^2y \oplus 2c_2xy^2 \oplus (c_2+e_2)y^3,$$

from which it follows that a polynomial $x^3 \oplus c_3xy^2 \oplus (c_2+e_2)y^3$, where $c_3 = \widehat{\min}\{e_2, 2c_2\}$, is in J. Applying Lemma 6.1.1 to

$$(-c_2-e_2)x^3 \oplus (c_3-c_2-e_2)xy^2 \oplus y^3 \quad \text{and}$$
$$(c_3-c_2-2e_2)x^3 \oplus (c_3-2e_2)x^2y \oplus (c_3-c_2-e_2)xy^2$$

we get that the polynomial

$$a_3x^3 \oplus (c_3-2e_2)x^2y \oplus y^3$$

is in J and $b_3 = c_3 - 2e_2$.

If $c_2 = \infty$ then note that $a_3 = \infty = c_2 - 2e_2$ $b_3 = -e_2$. But we also have $c_3 = e_2$ so $b_3 = c_3 - 2e_2$.

Now we will proceed by induction. So let us assume that $a_d = c_{d-1}-(d-1)e_2$ and $b_d = c_d - (d-1)e_2$ for all $d < i$, for some i. First assume $c_i = \infty$. This implies $b_i = \infty$ so we indeed have that $b_i = c_i - (d-1)e_2$. If $c_i = \infty$ we must have $e_i \neq \infty$ and using Lemma 6.1.3 we get

$$a_i = -e_i = -e_2 - c_{d-1}.$$

Note that since $c_i = \infty$, we have $c_{i-1} \neq \infty$ and so $a_{i-1} = c_{i-i} - (d-1)e_2 \neq \infty$. Let us consider the polynomials

$$a_{i-1}x^i \oplus b_{i-1}x^{i-1}y \oplus xy^{i-1} \quad \text{and}$$
$$a_{i-1}x^i \oplus (a_{i-1}+e_i)y^i.$$

Using Lemma 6.1.1 to eliminate x^i and dividing the resulting polynomial by y we get that

$$b_{i-1}x^{i-1} \oplus xy^{i-2} \oplus (a_{i-1}+e_i)y^{i-1}$$

is in J. From this it follows that $c_{i-1} = -b_{d-1}$. So by induction we have

$$a_i = -e_2 + b_{i-1} = c_{i-1} - (i-1)e_2.$$

Assume now that $c_i, e_i \neq \infty$. It follows from Lemma 6.1.2 that $a_i x^i \oplus b_i x^{i-1} y \oplus y^i$ is in J for some $a_i, b_i \in \mathbb{R}$. So both polynomials

$$\begin{aligned} (a_i + e_i - c_i)x^i \oplus (b_i + e_i - c_i)x^{i-1}y \oplus (e_i - c_i)y^i \quad &\text{and} \\ (-c_i)x^i \oplus xy^{i-1} \oplus (e_i - c_i)y^i \end{aligned}$$

are in J for some $c_i \in \mathbb{R}$. We use Lemma 6.1.1 on these two polynomials to eliminate y^i and after dividing the resulting equation by x we get that

$$a_{i-1}x^{i-1} \oplus (b_i + e_i - c_i)x^{i-2}y \oplus y^{i-1}$$

is in J. So $b_{i-1} = b_i + e_i - c_i$. Using Lemma 6.1.3 we get $b_{i-1} = b_i + e_2 + c_{i-1} - c_i$ so $b_i = b_{i-1} - e_2 - c_{i-1} + c_i$ and by induction

$$b_i = c_{i-1} - (i-2)e_2 - e_2 - c_{i-1} + c_i = c_i - (i-1)e_2.$$

To get the expression for a_i consider the polynomials

$$\begin{aligned} (a_i + e_{i-1} - c_{i-1})x^i \oplus (b_i + e_{i-1} - c_{i-1})x^{i-1}y \oplus (e_{i-1} - c_{i-1})y^i \quad &\text{and} \\ (-c_{i-1})x^{i-1}y \oplus xy^{i-1} \oplus (e_{i-1} - c_{i-1})y^i. \end{aligned}$$

Eliminating y^i and dividing by x we get that

$$(a_i + e_{i-1} - c_{i-1})x^{i-1} \oplus b_{i-1}x^{i-2}y \oplus y^{i-1}$$

is in J. So we have that

$$a_{i-1} = a_i + e_{i-1} - c_{i-1} = a_i + e_2 + c_{i-2} - c_{i-1}.$$

By induction we get

$$a_i = c_{i-2} - (i-2)e_2 - e_2 - c_{i-2} + c_{i-1} = c_{i-1} - (i-1)e_2.$$

□

Remember that in the case of the ideals in two variables without coefficients, we could describe the whole ideal by the lowest degree which contains a binomial. In the case of ideals with coefficients this information is not enough. However, if we know the ideal up to the degree where the first binomial appears, we know the whole ideal.

Proposition 6.1.5. *Let J be an ideal in $\mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ and assume that J contains a binomial. Let d be the smallest degree such that a polynomial p with support $\{x^d, y^d\}$ is in J. Then the polynomials of minimal support up to degree $d-1$ together with p determine uniquely the whole ideal.*

Proof. The only polynomial of minimal support in J_d which contains both x^d and y^d is p. All other polynomials of minimal support are of the form xp_1 or yp_2 for some homogeneous polynomials p_1, p_2 of degree $d-1$. Since the ideal is saturated we know the coefficients in these polynomials. From Proposition 5.1.1, for every degree m, we know all (non-valuated) circuits in J_m. For a given degree m, by saturation, we also know all valuated circuits which do not contain x^m and y^m at the same time. Let us assume that $m > d$ is not divisible by d and consider a circuit with support $\{x^m, x^{m-j}y^j, y^m\}$, where $1 \leq j \leq m-1$. Note that this set is a circuit if and only if j, m and $m-j$ are not divisible by d. Otherwise a two element subset of these monomials would be a binomial in J. To determine coefficients in the corresponding polynomial we consider two circuits

$$\{x^m, x^d y^{m-d}, x^{m-j}y^j\} \quad \text{and} \quad \{x^d y^{m-d}, y^m\}. \tag{6.2}$$

Note that $\{x^m, x^d y^{m-d}, x^{m-j}y^j\}$ must be a circuit since if we look at any two monomials from this set, after reducing the polynomial they form to the one with support of the form $\{x^i, y^i\}$, we get that i is not divisible by d. Using saturation we know the coefficients in the polynomials in J corresponding to circuits in (6.2). So we can use Lemma 6.1.1 to get the coefficients in the polynomial having support $\{x^m, x^{m-j}y^j, y^m\}$. Let us assume now that m is divisible by d. Then $\{x^m, y^m\}$ is a circuit and there are no other circuits in degree m containing both these two monomials. We know the coefficients in the polynomials with supports $\{x^m, x^{m-d}y^d\}$ and $\{x^{m-d}y^d, y^m\}$ and using Lemma 6.1.1 we get the coefficients in the polynomials with support $\{x^m, y^m\}$. $\square$

If we know an ideal $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ up to degree d for some d and J does not have any binomials up to this degree then we cannot determine from this the whole ideal. However if we are given a polynomial in J_{d+1} with the support in the set $\{x^{d+1}, xy^d, y^{d+1}\}$, we can determine what J_{d+1} is.

Proposition 6.1.6. *Let J be an ideal in $\mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ such that J does not have a circuit of size two for any degree less or equal to d and let $p = x^d \oplus c_d xy^{d-1} \oplus e_d y^d \in J$, where $c_d, e_d \in \mathbb{R}$. Then the degree d part of J, J_d, is determined by $\cup_{d'<d} J_{d'}$ and the polynomial p.*

Proof. Since J is saturated we know the coefficients in the polynomials which do not contain both x^d and y^d at the same time. To determine coefficients α_0, β_0 in a polynomial $p_0 = x^d \oplus \alpha_0 x^{d-j}y^j \oplus \beta_0 y^d$, where $1 \leq j < d-1$, let us consider polynomials p, $p_1 = x^d \oplus \alpha_1 x^{d-j}y^j \oplus \beta_1 xy^{d-1}$ and $p_2 = x^{d-j}y^j \oplus \alpha_2 xy^{d-1} \oplus \beta_2 y^d$ in J, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Using Lemma 6.1.1 to eliminate xy^{d-1} in the polynomials

$$\beta_1 p = \beta_1 x^d \oplus (\beta_1 + c_d)xy^{d-1} \oplus (\beta_1 + e_d)y^d \quad \text{and}$$
$$c_d p_1 = c_d x^d \oplus (c_d + \alpha_1)x^{d-j}y^j \oplus (c_d + \beta_1)xy^{d-1}$$

we get the coefficients in front of $x^{d-j}y^j$ and y^d and a set of possible coefficients for x^d in the polynomial p. Similarly, by eliminating xy^{d-1} in the polynomials

$$\alpha_2 p = \alpha_2 x^d \oplus (\alpha_2 + c_d)xy^{d-1} \oplus (\alpha_2 + e_d)y^d \quad \text{and}$$
$$c_d p_2 = c_d x^{d-j}y^j \oplus (c_d + \alpha_2)xy^{d-1} \oplus (c_d + \beta_2)y^d$$

we get the coefficients in front of x^d and $x^{d-j}y^j$ and a set of coefficients for y^d in p. We can now use a coefficient in front of $x^{d-j}y^j$ to determine the remaining two ones. $\square$

It turns out that every ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$ corresponds to either a finite or an infinite sequence of numbers in $\bar{\mathbb{R}}$.

Theorem 6.1.7. *Let J be an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$. For each degree d, let c_d be the coefficient in front of the monomial xy^{d-1} in the polynomial $x^d \oplus c_d xy^{d-1} \oplus e_d y^d$, where $e_d \in \bar{\mathbb{R}}$. Then J can be described uniquely by a sequence $\{c_d\}_{d=2}^{\infty}$. If for some d' we have $c_{d'} = \infty$ then the sequence $\{c_d\}_{d=2}^{d'}$ describes the ideal J completely.*

Proof. If an ideal J does not have any binomials then $x^d \oplus c_d xy^{d-1} \oplus e_d y^d$, where $c_d, e_d \in \mathbb{R}$, is of minimal support for any d and the statement follows from Proposition 6.1.6 and Lemma 6.1.3. If d is the smallest degree such that a polynomial with support $\{x^d, y^d\}$ is in J then by Proposition 6.1.6 the elements of the sequence up to c_{d-1} describe the ideal up to degree $d-1$. We have $c_d = \infty$ so by Proposition 6.1.5 and Lemma 6.1.3 J is also determined for all degrees greater than $d-1$. In particular, this means that for every degree d there is a unique polynomial whose support is a subset of $\{x^d, xy^{d-1}, y^d\}$ and whose coefficient in front of x^d is 0. The coefficients in front of xy^{d-1} give the remaining entries in the sequence. $\square$

Not every sequence of numbers in $\bar{\mathbb{R}}$ gives us an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$. The rest of this section is devoted to studying which sequences gives us such tropical ideals.

Lemma 6.1.8. *Let $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ with J_2 being generated by $\{x^2 \oplus c_2xy \oplus e_2y^2\}$, for some $c_2 \in \bar{\mathbb{R}}$, $e_2 \in \mathbb{R}$. For any $d \geq 3$, let c_d and e_d be coefficients in front of xy^{d-1} and y^d, respectively, in the polynomial in J whose support is a subset of $\{x^d, xy^{d-1}, y^d\}$ and whose coefficient in front of x^d is 0. Then $c_d = \widehat{\min}\{e_2 + c_{d-2}, c_2 + c_{d-1}\}$, where we define $c_1 := 0$.*

Proof. First assume $c_{d-1} \neq \infty$. Using Lemma 6.1.1 on the polynomials

$$x^d \oplus c_{d-1}x^2y^{d-2} \oplus e_{d-1}xy^{d-1} \quad \text{and}$$
$$c_{d-1}x^2y^{d-2} \oplus (c_2 + c_{d-1})xy^{d-1} \oplus (e_2 + c_{d-1})y^d$$

in J, we get that

$$x^d \oplus c_dxy^{d-1} \oplus (e_2 + c_{d-1})y^d$$

is in J, where

$$c_d = \widehat{\min}\{e_{d-1}, c_2 + c_{d-1}\}.$$

From Lemma 6.1.3 we have that $e_{d-1} = e_2 + c_{d-2}$ so

$$c_d = \widehat{\min}\{e_2 + c_{d-2}, c_2 + c_{d-1}\}$$

indeed.

If $c_{d-1} = \infty$ then $c_d = e_{d-1}$ and so by Lemma 6.1.3 $c_d = e_2 + c_{d-2}$. Since $c_{d-1} = \infty$, we get that $c_d = \widehat{\min}\{e_2 + c_{d-2}, c_2 + c_{d-1}\}$. $\square$

Lemma 6.1.9. *Let $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ with J_2 being generated by $\{x^2 \oplus c_2xy \oplus e_2y^2\}$, for some $c_2 \in \bar{\mathbb{R}}$, $e_2 \in \mathbb{R}$. Then the following holds:*

1. *If $e_2 > 2c_2$ then, for any $d \geq 3$, $c_d = (d-1)c_2$, $e_d = e_2 + (d-2)c_2$. This implies that J is determined by c_2 and e_2 whenever $e_2 > 2c_2$.*

2. *If $e_2 < 2c_2$ then, for any $d \geq 1$, $c_{2d+1} = de_2$ and $c_{2d} \geq (d-1)e_2 + c_2$.*

Proof. 1. First we will prove by induction that $c_d = (d-1)c_2$. Since $e_2 > 2c_2$, by Lemma 6.1.8 we get that $c_3 = 2c_2$. Assume that $c_d = (d-1)c_2$ for all $3 \leq d \leq i$ for some integer i. Then, using Lemma 6.1.8,

$$\begin{aligned} c_{i+1} &= \widehat{\min}\{e_2 + c_{i-1}, c_2 + c_i\} \\ &= \widehat{\min}\{e_2 + (i-2)c_2, ic_2\} \\ &= ic_2, \end{aligned}$$

where the last equality is due to the fact that $e_2 > 2c_2$.

From Lemma 6.1.3, $e_d = e_2 + c_{d-1}$. But since we proved that $c_{d-1} = (d-2)c_2$ we get $e_d = e_2 + (d-2)c_2$.

The uniqueness of J follows from Theorem 6.1.7.

2. We will prove this by induction. From Lemma 6.1.8, $c_3 = e_2$ and $c_4 = \widehat{\min}\{e_2 + c_2, c_2 + c_3\} \geq e_2 + c_2$. Let us assume that $c_{2d} \geq (d-1)e_2 + c_2$ and $c_{2d+1} = de_2$ for all $d \leq i$ for some natural i. Then by Lemma 6.1.8

$$\begin{aligned} c_{2(i+1)} &= \widehat{\min}\{e_2 + c_{2i}, c_2 + c_{2i+1}\} \\ &= \widehat{\min}\{e_2 + c_{2i}, c_2 + ie_2\} \\ &\geq c_2 + ie_2, \end{aligned}$$

since $c_{2i+1} = ie_2$ and $c_{2i} \geq (i-1)e_2 + c_2$ by induction, and

$$\begin{aligned} c_{2(i+1)+1} &= \widehat{\min}\{e_2 + c_{2i+1}, c_2 + c_{2(i+1)}\} \\ &= \widehat{\min}\{e_2 + ie_2, c_2 + c_{2(i+1)}\}, \end{aligned}$$

since $c_{2i+1} = ie_2$. Let us consider $\widehat{\min}\{e_2 + ie_2, c_2 + c_{2(i+1)}\}$. Note that by induction $c_{2i} \geq (i-1)e_2 + c_2$ so $e_2 + c_{2i} \geq c_2 + ie_2$. It follows that

$$\begin{aligned} c_2 + c_{2(i+1)} &= c_2 + \widehat{\min}\{e_2 + c_{2i}, c_2 + c_{2i+1}\} \\ &= c_2 + \widehat{\min}\{e_2 + c_{2i}, c_2 + ie_2\} \\ &\geq c_2 + c_2 + ie_2 \\ &= 2c_2 + ie_2. \end{aligned}$$

Now since $e_2 < 2c_2$ we have

$$e_2 + ie_2 < 2c_2 + ie_2 \leq c_2 + c_{2(i+1)}$$

and it follows that $c_{2(i+1)+1} = e_2 + ie_2$.

□

Proposition 6.1.10. *Let $J \in \mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ with J_2 being generated by $\{x^2 \oplus c_2xy \oplus e_2y^2\}$, for some $c_2 \in \bar{\mathbb{R}}$, $e_2 \in \mathbb{R}$. Let $e_2 = 2c_2$ and define $c_1 := 0$. Then for any $i \geq 2$ the following is true:*

1. *For each i, $c_i \geq (i-1)c_2$.*

2. *If $c_i > (i-1)c_2$, then $c_{i+1} = ic_2$ and $c_{i+2} = (i+1)c_2$.*

3. *If* $c_i > (i-1)c_2$ *and* $c_i \neq \infty$ *then* $c_{ki} > (ki-1)c_2$ *for any* $k \in \mathbb{N}_+$. *If* $c_i = \infty$ *then* $c_{ki} = \infty$.

4. *If* $c_i > (i-1)c_2$ *then* $c_k = (k-1)c_2$ *whenever* $\gcd\{i,k\} \in \{1,2\}$.

Proof. 1. Since the polynomials

$$\begin{aligned} x^3 \oplus & c_2x^2y \oplus 2c_2xy^2 \quad \text{and} \\ & c_2x^2y \oplus 2c_2xy^2 \oplus 3c_2y^3 \end{aligned}$$

are in J, we have that the polynomial

$$x^3 \oplus c_3xy^2 \oplus 3c_2y^3$$

is in J, where $c_3 \geq 2c_2$. Assume that for some $l \geq 3$ we have $c_{\hat{l}} \geq (\hat{l}-1)c_2$ for all $\hat{l} \leq l$. Then by Lemma 6.1.8 we have

$$c_{l+1} = \widehat{\min}\{2c_2 + c_{l-1}, c_2 + c_l\}.$$

By induction $2c_2 + c_{l-1} \geq lc_2$ and $c_2 + c_l \geq lc_2$, so $c_{l+1} \geq lc_2$.

2. We have $c_i > (i-1)c_2$. First let us assume that $c_i \neq \infty$. The polynomials

$$\begin{aligned} x^{i+1} \oplus & c_ix^2y^{i-1} \oplus e_ixy^i \quad \text{and} \\ & c_ix^2y^{i-1} \oplus (c_i + c_2)xy^i \oplus (c_i + 2c_2)y^{i+1} \end{aligned}$$

are in J so the polynomial

$$x^{i+1} \oplus c_{i+1}xy^i \oplus (c_i + 2c_2)y^{i+1}$$

is also in J, where $c_{i+1} = \widehat{\min}\{e_i, c_i + c_2\} = \widehat{\min}\{2c_2 + c_{i-1}, c_i + c_2\}$. For a contradiction, let us assume that $c_{i+1} > ic_2$. By assumption $c_i + c_2 > ic_2$ so we must have $2c_2 + c_{i-1} > ic_2$, that is, $c_{i-1} > (i-2)c_2$. By induction this implies $c_2 > c_2$, a contradiction. So $c_{i+1} = ic_2$.

Let us now show that $c_{i+2} = (i+1)c_2$. We know that the polynomials

$$\begin{aligned} x^{i+2} \oplus & (ic_2)x^2y^i \oplus e_{i+1}xy^{i+1} \quad \text{and} \\ & (ic_2)x^2y^i \oplus (ic_2 + c_2)xy^{i+1} \oplus (ic_2 + 2c_2)y^{i+2} \end{aligned}$$

are in J, for some $e_{i+1} \in \bar{\mathbb{R}}$. This implies

$$x^{i+2} \oplus c_{i+2}xy^{i+1} \oplus (ic_2 + 2c_2)y^{i+2}$$

is in J, where $c_{i+2} = \widehat{\min}\{e_{i+1}, (i+1)c_2\} = \widehat{\min}\{2c_2 + c_i, (i+1)c_2\}$. Since $c_i > (i-1)c_2$ we get $c_{i+2} = (i+1)c_2$.

Consider now the case when $c_i = \infty$. Then we must have $c_{i-1} \neq \infty$. Assume that $c_{i-1} > (i-2)c_2$. Since $c_i = \infty$ we must have $c_{i-1} \neq \infty$. So we have $c_2 \neq \infty$ and by the first part of the proof of part 2 this implies $c_i = (i-1)c_2 \neq \infty$, a contradiction. So we must have $c_{i-1} = (i-2)c_2$. We also have $c_{i+1} = e_i = 2c_2 + c_{i-1} = ic_2$ so the polynomials

$$\begin{aligned} x^{i+2} \oplus & (ic_2)x^2y^i \quad \text{and} \\ & (ic_2)x^2y^i \oplus (ic_2 + c_2)xy^{i+1} \oplus (ic_2 + 2c_2)y^{i+2} \end{aligned}$$

are in J which implies that

$$x^{i+2} \oplus (ic_2 + c_2)xy^{i+1} \oplus (ic_2 + 2c_2)y^{i+2}$$

is in J and $c_{i+2} = (i+1)c_2$.

3. We will prove this by induction on k. First let us assume $c_i \neq \infty$. We have $c_i > (i-1)c_2$. Assume $c_{li} > (li-1)c_2$ for some l and $c_{li} \neq \infty$. We will show that $c_{(l+1)i} > ((l+1)i-1)c_2$. The polynomial $x^{li} \oplus c_{li}xy^{li-1} \oplus e_{li}y^{li}$ is in J, for some $c_{li}, e_{li} \in \bar{\mathbb{R}}$. From Lemma 6.1.4 it follows $(c_i - 2ic_2)x^{i+1} \oplus (c_{i+1} - 2ic_2)x^iy \oplus y^{i+1}$ is in J. So the polynomials

$$\begin{aligned} c_ix^{(l+1)i} \oplus & (c_i + c_{li})x^{i+1}y^{li-1} \oplus (c_i + e_{li})x^iy^{li} \quad \text{and} \\ & (c_{li} + c_i)x^{i+1}y^{li-1} \oplus (c_{li} + c_{i+1})x^iy^{li} \oplus (c_{li} + 2ic_2)y^{(l+1)i} \end{aligned}$$

are also in J and eliminating $x^{i+1}y^{li-1}$ we get that

$$c_ix^{(l+1)i} \oplus wx^iy^{li} \oplus (c_{li} + 2ic_2)y^{(l+1)i}$$

is in J, where $w = \widehat{\min}\{e_{li} + c_i, c_{i+1} + c_{li}\}$. But since $c_i > (i-1)c_2$ and $c_{li} > (li-1)c_2$ we have $e_{li} + c_i = 2c_2 + c_{li-1} + c_i > 2c_2 + (li-2)c_2 + (i-1)c_2 = ((l+1)i-1)c_2$ and $c_{i+1} + c_{li} > ic_2 + (li-1)c_2 = ((l+1)i-1)c_2$. So

$w > ((l+1)i-1)c_2$. Moreover the polynomial

$$wx^i y^{li} \oplus (c_i + w)xy^{(l+1)i-1} \oplus (e_i + w)y^{(l+1)i}$$

is in J so

$$c_i x^{(l+1)i} \oplus (c_i + w)xy^{(l+1)i-1} \oplus e_{(l+1)i} y^{(l+1)i}$$

is in J for some $e_{(l+1)i} \in \bar{\mathbb{R}}$. It follows that

$$\begin{aligned} c_{(l+1)i} &= c_i + w - c_i \\ &= w \\ &> ((l+1)i-1)c_2. \end{aligned}$$

If $c_{li} = \infty$, then the polynomials

$$x^{(l+1)i} \oplus e_{li} x^i y^{li} \qquad \text{and} \qquad a_{i+1} x^{i+1} y^{li-1} \oplus b_{i+1} x^i y^{li} \oplus y^{li+i}$$

are in J for some $e_{li}, a_{i+1} \in \mathbb{R}$. By Lemma 6.1.4 and part 2, we have

$$a_{i+1} = c_i - 2ic_2$$

and

$$b_{i+1} = c_{i+1} - 2ic_2 = -ic_2.$$

It follows that the polynomials

$$\begin{gathered} -(l+1)ic_2 x^{(l+1)i} \oplus (-ic_2)x^i y^{li} \quad \text{and} \\ (c_i - 2ic_2)x^{i+1} y^{li-1} \oplus (-ic_2)x^i y^{li} \oplus y^{li+i} \end{gathered}$$

are also in J. By eliminating $x^i y^{li}$ we get that

$$-(l+1)ic_2 x^{(l+1)i} \oplus (c_i - 2ic_2)x^{i+1} y^{li-1} \oplus y^{li+i}$$

is in J. The polynomial $x^{i+1} \oplus c_{i+1} xy^i \oplus e_{i+1} y^{i+1}$ is also in J, where, by Lemma 6.1.3 and part 2, we have $c_{i+1} = ic_2$ and $e_{i+1} = 2c_2 + c_i$. So we can eliminate $x^{i+1}y^{li-1}$ from the polynomials

$$\begin{aligned}-(l+1)ic_2x^{(l+1)i} \oplus (c_i - 2ic_2)x^{i+1}y^{li-1} &\oplus y^{(l+1)i}\\ \text{and}\quad (c_i - 2ic_2)x^{i+1}y^{li-1} &\oplus (c_i - ic_2)xy^{(l+1)i-1}\\ &\oplus (2c_i + (2-2i)c_2)y^{(l+1)i}\end{aligned}$$

to get that

$$-(l+1)ic_2x^{(l+1)i} \oplus (c_i - ic_2)xy^{(l+1)i-1} \oplus (e_{(l+1)i} - (l+1)ic_2)y^{(l+1)i}$$

is in J for some $e_{(l+1)i}$. So we get that

$$\begin{aligned}c_{(l+1)i} &= c_i - ic_2 + (l+1)ic_2\\ &= c_i + lic_2\\ &> (i-1)c_2 + lic_2\\ &= ((l+1)i-1)c_2.\end{aligned}$$

Now assume $c_i = \infty$. This means a polynomial with support $\{x^i, y^i\}$ is in J. It follows from the proof of Theorem 4.2.4 that for any k a polynomial with support x^{ki}, y^{ki} is in J. In particular this means that $c_{ki} = \infty$.

4. First let us assume that $\gcd\{i,k\} = 2$. There exists some $s, t \in \mathbb{N}$ such that $tk - si = \pm 2$. Without loss of generality let us assume that s and t are such that $tk = si + 2$. Since $c_i > (i-1)c_2$, by part 3 we have that $c_{si} > (si-1)c_2$ (or $c_{si} = \infty$ if $c_i = \infty$), and by part 2 we have that $c_{si+2} = (si+1)c_2$. Assume that $c_k > (k-1)c_2$. Note that we cannot have $c_k = \infty$ as by part 2, since $c_{si} > (si-1)c_2$ we have $c_{tk} = c_{si+2} = (tk-1)c_2$. Then we have $c_{si+2} = c_{tk} > (tk-1)c_2 = (si+1)c_2$, a contradiction. The proof when $\gcd i, k = 1$ is exactly the same.

$\square$

For each $c_i \geq (i-1)c_2$, let us set $s_i = c_i - (i-1)c_2$. The following lemma and two conjectures give the sufficient conditions for a sequence of numbers in $\bar{\mathbb{R}}$ to correspond to an ideal in $\mathcal{I}^{\mathrm{tr},2,\bar{\mathbb{R}}}_{\mathrm{x,y}}$.

Lemma 6.1.11. *Let $e_2 = 2c_2$. Then for any $i \geq 3$, if $s_i > 0$ then $s_{ki} \geq s_i$ for all $k \in \mathbb{N}_+$.*

Proof. The proof is almost exactly the same as the proof of part 3 in Proposition 6.1.10, so here we only indicate the differences.

In the proof of part 3 in Proposition 6.1.10 we defined $w = \widehat{\min}\{e_{li}+c_i, c_{i+1}+c_{li}\}$. We also saw that if $c_i > (i-1)c_2$ and $c_{li} > (li-1)c_2$ then $c_{(l+1)i} = w$. We know $s_i > 0$, $s_{li} \geq 0$, $s_{i+1} = 0$ and it follows that $c_i > (i-1)c_2$ and, by Proposition 6.1.10, $c_{li} > (li-1)c_2$. So we have

$$\begin{aligned} c_{(l+1)i} &= \widehat{\min}\{e_{li} + c_i, c_{i+1} + c_{li}\} \\ &= \widehat{\min}\{2c_2 + c_{li-1} + s_i + (i-1)c_2, ic_2 + s_{li} + (li-1)c_2\} \\ &= \widehat{\min}\{c_{li-1} + s_i + (i+1)c_2, s_{li} + ((l+1)i-1)c_2\}. \end{aligned}$$

Also $c_{li-1}+s_i+(i+1)c_2 \geq ((l+1)i-1)c_2+s_i$ and $s_{li}+((l+1)i-1)c_2 \geq ((l+1)i-1)c_2+s_i$ so $c_{(l+1)i} \geq ((l+1)i-1)c_2 + s_i$ and it follows that $s_{(l+1)i} \geq s_i$. The result follows now by induction.

□

Conjecture 6.1.12. *Let $e_2 = 2c_2$ and let $s_i > 0$ for some i. Assume that k is the smallest natural number such that $s_{ki} > s_i$. Then for all j such that $ki < j < 2ki$, $s_j = s_{\gcd\{j,ki\}}$.*

6.2 Sufficient conditions

We claim that the conditions described in the previous section are in fact sufficient for a sequence of numbers in $\bar{\mathbb{R}}$ to correspond to a tropical ideal in $\mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$.

Conjecture 6.2.1. *Let $s = \{c_d\}_2^\infty$ be a sequence satisfying the conditions in Proposition 6.1.10, Lemma 6.1.11 and in Conjecture 6.1.12. Then there exists a unique ideal J in $\mathcal{I}_{\mathrm{x,y}}^{\mathrm{tr},2,\bar{\mathbb{R}}}$ whose degree two part is generated by the polynomial $x^2 \oplus c_2xy \oplus 2c_2y^2$ and such that for all degrees d, c_d is the coefficient in the polynomial $x^d \oplus c_dxy^{d-1} \oplus e_dy^d \in J_d$, for some $e_d \in \bar{\mathbb{R}}$.*

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